CHAPTER 17

Simple Harmonic Motion

We're now going to study what are called small oscillations, or simple harmonic motion. Take any mechanical system that is in a state of equilibrium. Equilibrium means the forces on the body add up to zero. It has no desire to move. If you give it a little kick, a push away from the equilibrium point, what will happen? There are two main possibilities. Imagine a marble on top of a hill. That is in *unstable equilibrium* because if you give the marble a nudge, it will roll downhill and never return to you. The other possibility involves *stable equilibrium*: if you push the system away from equilibrium, there are forces bringing it back. The standard example is a marble in a bowl: when it is shaken from its position at the bottom, it will rock back and forth until it settles again. A rod hanging vertically from the ceiling from a pivot, when pulled to the side and released, will swing back and forth. These are examples of simple harmonic motion, which results whenever any system is slightly disturbed from stable equilibrium.

The example that we're going to consider is a mass m, resting on a table, connected to a spring, which in turn is connected to the wall. The spring is not stretched or contracted; the mass is at rest, as shown in Figure 17.1. That's what I mean by equilibrium. Now let it be displaced by

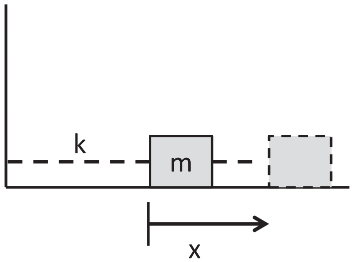

Figure 17.1 The mass m rests on a table and is connected to a spring of force constant k, which is anchored to the wall. The displacement from equilibrium is denoted by x. It is positive in the figure but it could also be negative if the mass were to be displaced the other way.

an amount x from this point of equilibrium. The spring force is $F = -kx$ and Newton's law says

$$m\frac{d^2x}{dt^2} = -kx. \tag{17.1}$$

If the mass strays to the right, x is positive and $-kx$ is to the left, so as to send it back toward its equilibrium position. If x is negative, the restoring force is positive, again pointing to the equilibrium position.

We want to understand the behavior of such a mass. How do we solve this problem? Our job is to find the function $x(t)$ that satisfies this equation, which we rewrite as follows:

$$\frac{d^2x}{dt^2} = -\omega^2 x \tag{17.2}$$

$$\omega = \sqrt{\frac{k}{m}}. \tag{17.3}$$

You can make it a word problem and say, "I'm looking for a function whose second derivative is minus itself, except for this number ω^2." Trigonometric functions have the property that if you take two derivatives, they return to minus themselves. So, you can guess that $x = \cos t$ but it won't work, as I showed you before. On the other hand, the guess

$$x(t) = A\cos\omega t \tag{17.4}$$

will obey this equation. While A is clearly the amplitude, ω is related to the frequency of oscillations as follows. If I start at $t = 0$ when $x = A$, how long do I have to wait until it comes back to A? I have to wait a time T, such that

$$\omega T = 2\pi \tag{17.5}$$

because that's when the cosine returns to 1. That means the time that I have to wait is

$$T = \frac{2\pi}{\omega}. \tag{17.6}$$

You can rewrite this as

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{17.7}$$

where $f = \frac{1}{T}$ is what we would normally call frequency, which is how many oscillations it completes per second. It is the inverse of the time period. In physics talk, frequency usually means ω.

So, if you pull a mass and let it go, it oscillates with a frequency that is connected to the force constant and the mass. If the spring is very stiff and k is very large, the frequency is very high. If the mass is very big and the motion is very sluggish, f is diminished. So, all that stuff you expect intuitively is quantified by the solution to the equation, but there is more. For example, it is not intuitively obvious that if you make the mass four times as big, you will double the time period.

One remarkable part of the solution is that you can pick any A you like without changing ω or T. Think about what that means. The amplitude A is the amount by which you pulled the mass when you let it go. You find that whether you pull the spring by one inch or by ten inches, it takes the same time to finish a full back-and-forth motion. If you pull it by two inches, compared to one inch, it has a longer way to go. But if you pull it by two inches, the spring is going to be that much more tense, and it's going to exert a bigger force so that it will go faster for most of the time; that's very clear. But the fact that it goes faster in exactly the right way to complete the trip in exactly the same time is rather a miraculous property of Eqn. 17.2. If you tamper with it, if you add to the force even a tiny extra term, say proportional to x^3, then this feature is gone. It's like saying that planets move around the sun in *closed* elliptical orbits only under the $\frac{1}{r^2}$ force. It is not true if the force falls as $\frac{1}{r^{2.0000001}}$.

Now consider the following variant of this solution. You set your clock to 0 at the origin in the graph. Suppose I set my clock to 0, right there on the dotted vertical line in Figure 17.2 at $t = \frac{\pi}{2}$. When my clock says 0, x is not at the maximum; it vanishes. But it's the same physics, and it's the same equation. Where, then, is the solution that describes what I see? It is there and it comes from the fact that we had the latitude of adding a certain angle ϕ, called a *phase*, to the solution:

$$x(t) = A\cos[\omega t + \phi]. \tag{17.8}$$

Your choice is $\phi = 0$ and mine is $\phi = \frac{\pi}{2}$. You can verify that whatever we pick for ϕ, the above $x(t)$ will be a solution because two derivatives of the solution with the ϕ is also $-\omega^2$ times itself. And, whatever you pick for A, it will still work, because A cancels out of both sides in Eqn. 17.2. So, whenever you have an oscillator, say, a mass and spring system, and you want to know what x is going to be at all times, it is not enough to know that it obeys Eqn. 17.2; you need to know the amplitude and the phase. These are determined by knowing two things about the solution, which is usually the x and v at some time, usually $t = 0$. For this reason we refer to them as *initial value data*.

Let me give you an example. Suppose an oscillator has $x(0) = 5$ and velocity $v(0) = 0$, at $t = 0$. What does that mean? I pulled the mass by 5 and I let it go. I give you the values of the spring constant k and the mass m, and I say, "What's the future x?"

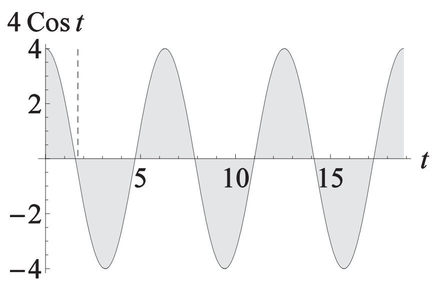

Figure 17.2 The function $A\cos\omega t$ for the case $A = 4$, $\omega = 1$. The dotted vertical line is another possible way to set the clock to zero, another choice of phase, namely $\phi = \frac{\pi}{2}$.

First observe that the velocity corresponding to our solution Eqn. 17.8 is

$$v(t) = -\omega A \sin(\omega t + \phi). \tag{17.9}$$

So we know two things at $t = 0$:

$$5 = A\cos[0 + \phi] \tag{17.10}$$
$$0 = -A\omega \sin[0 + \phi]. \tag{17.11}$$

The second equation gives us two choices: either $A = 0$, which is a trivial solution, or $\phi = 0$, which lets A survive this test. The first equation with $\phi = 0$ gives $A = 5$ leading to the solution $x(t) = 5\cos\omega t$. This is a problem where we did not need a non-zero ϕ. But it could have been that when you joined the experiment, you were somewhere to the right of the origin, on the vertical dotted line in Figure 17.2, when you set your clocks to zero. Then you would have, as your initial conditions, $x = 0, v = -5\omega$, which means $\phi = \frac{\pi}{2}$. Of course $A = 5$ as before.

Let us agree that, if there's only one oscillator, it is perverse to set your clock to 0 at any time other than when the oscillator is at its maximum displacement, so that

$$x(t) = A\cos\omega t. \tag{17.12}$$

(If there are two oscillators oscillating out of step, it's impossible to make $\phi = 0$ for both of them: you can set your clock to 0 when one of them is at a maximum, but then the other may not be at its maximum.) Going forward, remember that the velocity and acceleration are, for all future times,

$$v(t) = -\omega A \sin\omega t \tag{17.13}$$
$$a(t) = -\omega^2 A\cos\omega t \quad \left(= -\frac{k}{m}x(t) \quad \text{in accordance with } F = ma\right). \tag{17.14}$$

So the velocity also oscillates sinusoidally but with an amplitude ωA. The acceleration also oscillates but with an amplitude $\omega^2 A$. These two results are true for any phase ϕ.

Let us explicitly verify the law of conservation of energy. Consider the total energy:

$$E(t) = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \tag{17.15}$$

$$= \frac{1}{2}m\omega^2 A^2 \sin^2 \omega t + \frac{1}{2}kA^2 \cos^2 \omega t \tag{17.16}$$

$$= \frac{1}{2}kA^2 \quad \text{because } \omega^2 = \frac{k}{m}. \tag{17.17}$$

Thus, by magic, the time-dependent terms $\sin^2 \omega t$ and $\cos^2 \omega t$ have the same coefficient, and you find that $E(t)$ actually does not depend on time at all. Even though position and velocity are constantly changing, this combination will not depend on time. At the instant when the mass has reached one extremity and is about to swing back, it has no velocity; it only has an $x = A$, and the energy of the oscillator is all potential energy, $\frac{1}{2}kA^2$.

17.1 More examples of oscillations

If a body is in stable equilibrium, and you disturb it, it rocks back and forth, executing simple harmonic motion. The standard textbook example is the mass on a spring, which we just studied. But it is a very generic situation, as shown in Figure 17.3. Skipping the mass-and-spring example, let us go to the top right, where we have a beam hanging from the ceiling by a cable that is fixed to its center of mass (CM). If you twist it by an angle θ, it will try to untwist itself. Now we don't have a restoring force but we have a restoring torque. What can be the expression for the restoring torque τ? When you don't do anything, the cable doesn't do anything, so τ vanishes when $\theta = 0$. If $\theta \neq 0$, it is some function of θ, and the leading term in the Taylor expansion would be proportional to θ:

$$\tau(\theta) = -\kappa\theta. \tag{17.18}$$

The coefficient κ is the *torsion constant*, and the minus sign tells you it's a restoring torque. That means if you make θ positive, the torque will try to twist you the other way. The torsion constant, which is the restoring torque per unit angular displacement, is to rotations what the spring constant was to linear oscillations: the restoring force per unit displacement.

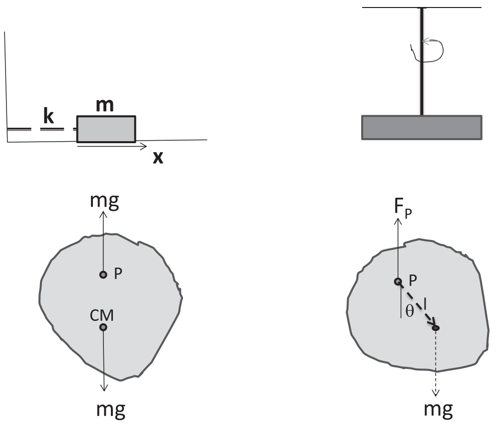

Figure 17.3 Some examples of simple harmonic motion. The left shows the eternal favorite, the mass and spring; top right is a beam hanging from the ceiling by a cable; the bottom left and right show a physical pendulum supported at the pivot P, when it is in equilibrium, and when it is displaced by an angle θ. The vector $\mathbf{g}$ represents the downward gravitational field of magnitude $9.8m/s^2$.

You have to find this κ, which is not given, the way k is. Once you do, you can say

$$I\frac{d^2\theta}{dt^2} = -\kappa\theta \tag{17.19}$$

where I is the moment of inertia of the beam about the point of suspension.

Mathematically, this equation is identical in form to

$$m\frac{d^2x}{dt^2} = -kx \tag{17.20}$$

with the substitution $x \to \theta, m \to I$, $k \to \kappa$. So the answer follows:

$$\theta(t) = A\cos\omega t \tag{17.21}$$

$$\omega = \sqrt{\frac{\kappa}{I}}. \tag{17.22}$$

The mass-spring system executes linear oscillations, while the beam executes angular oscillations. Another example of the latter is the simple pendulum. The pendulum has a bob of mass m hanging by a massless rod of length l. If you let it hang vertically, it will stay that way forever. No torque, no motion. Suppose you pull it by an angle θ and release it. To predict the future, you need to find I and κ. Now I is easy: for a single mass m at a distance l from the pivot point, $I = ml^2$. To find κ, you need to find the restoring torque per angular displacement. If you displace by θ, the torque about the pivot point is

$$\tau = -mgl\sin\theta \simeq -mgl\theta \tag{17.23}$$

where I have approximated $\sin\theta$ by θ, which is the leading term in the Taylor expansion. With just this term, we can read off κ:

$$\kappa = -\frac{\tau}{\theta} = mgl. \tag{17.24}$$

So

$$\omega = \sqrt{\frac{\kappa}{I}} = \sqrt{\frac{mgl}{ml^2}} = \sqrt{\frac{g}{l}}, \tag{17.25}$$

from which follows the familiar formula

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}. \tag{17.26}$$

Notice that if you displace the pendulum by large angles, when $\sin\theta$ cannot be approximated by θ, the frequency will no longer be independent of the amplitude.

Note that finding ω took some work. You had to disturb the system from equilibrium and find the restoring torque per unit angle $\kappa = -\frac{\tau}{\theta}$ and also compute I, whereas in the case of the mass-spring system, you were simply given m and k. In the case of the twisted cable, κ will be given to you, because computing it from first principles requires work beyond the scope of this course.

Let us move from a pendulum with all the mass concentrated in the bob to a *physical pendulum*, some irregularly shaped flat planar object, as shown in the middle of the second line of Figure 17.3. You drive a nail through it at some pivot point P and hang it on the wall. It will come to rest in a certain equilibrium configuration. Think about where the center of mass will be. It will lie somewhere on the vertical line going through P—otherwise the force of gravity, which is effectively acting at the CM, will produce a torque around P.

Let us look at the forces. This body, when hanging in its rest position, has two forces on it: the nail, which is pushing up, and the weight of the body $m\mathbf{g}$, which is pushing down, cancel each other. The nail will keep it from falling. The nail will not keep it from swinging, because the force of the nail, acting as it does at the pivot point, is unable to exert a torque, whereas the minute you rotate the body, $m\mathbf{g}$ is able to exert a torque, as is clear from the figure. That's why if you rotate it and let go, it will start swinging back and forth.

What will the torque be? It will be the same as before: $-mgl\sin\theta$, where l is now the distance between the pivot point and the center of mass. As far as the torque is concerned, it's as if all the mass were sitting at the CM. But the moment of inertia is not as if all the mass is sitting at the CM, in which case it would be ml^2. So don't make that mistake. All the mass is not sitting at the CM; it is all over the place. The moment of inertia is $I = I_{CM} + ml^2$ by the parallel axis theorem, where I_{CM} is hard to compute for an irregular object.

So, every problem that you will ever get will look like one of these two. Either something is moving linearly with a coordinate that you can call x, or something is rotating or twisting by an angle you can call θ. And if you want to find out the frequency of vibration, you have to disturb it from equilibrium—by pulling the mass, twisting the cable, or displacing the pendulum from its equilibrium position—in order to find the restoring force or torque per unit displacement.

17.2 Superposition of solutions

I will now go over more complicated oscillations using some of the formulas we learned in the last chapter. Here is the most important one:

$$e^{i\theta} = \cos\theta + i\sin\theta. \qquad (17.27)$$

This is a formula worth memorizing. You should realize that given any expression involving complex numbers, you can get another equation by taking the complex conjugate of both sides, where every i is changed to minus i. That will give you

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{17.28}$$

This is true because if two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal, then the real and imaginary parts are separately equal, and so are their complex conjugates: $z_1^* = x_1 - iy_1 = x_2 - iy_2 = z_2^*$. The two previous equations can be inverted to give

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{17.29}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{17.30}$$

You don't need trigonometric functions once you have the exponential function, provided you let the exponent be complex or imaginary. This is one example of unification. People always say Maxwell unified this, and Einstein tried to unify that. Unification means things that you thought were unrelated are, in fact, related, and they are different manifestations of the same thing. When we first discovered trigonometric functions, we were thinking right-angle triangles, opposite sides and adjacent sides, and so on. Then, we discovered the exponential function, which, by the way, was used by bankers who were trying to calculate compound interest continuously at every instant. The fact that those functions are related is a marvelous result, but it emerges only if you invoke complex numbers.

Finally, remember that there are two ways to write a complex number:

$$z = x + iy = re^{i\theta} \equiv |z|e^{i\theta}. \tag{17.31}$$

Now we use the new tools to attack the familiar equation

$$\ddot{x} \equiv \frac{d^2x}{dt^2} = -\omega_0^2 x \qquad \omega_0 = \sqrt{\frac{k}{m}} \tag{17.32}$$

where the second derivative of x is written as $\ddot{x}$, and ω_0, the natural frequency of vibrations of the oscillator, has been given a subscript to distinguish it from other ω's that will arise shortly. Earlier we solved this equation by turning it into a word problem: "What is the function, $x(t)$, with the property that two derivatives of the function look like the function itself, except for a proportionality constant?" We racked our brains and we remembered that sines and cosines had this property. One derivative is no good; it turns sine into cosine and vice versa. Two derivatives bring back the function you started with, which is why the answer could be sines or cosines. But now I'm going to solve the equation a different way. I know a function that is even better—it reproduces itself when it is differentiated *once*. If so, it's obvious that its 92-nd derivative will also look like the function. But recall why we rejected

$$x(t) = Ae^{t}. \tag{17.33}$$

I want to get something proportional to $-x(t)$ upon taking two derivatives, and this does not do it: I get $+x(t)$. It does not help to try something like Ae^{-t} because after two derivatives I again get $+x(t)$. So this function is no good. Also, it doesn't look like what I want. Even without doing much work, I know that if I pull this spring it's going to go back and forth, whereas these functions are exponentially growing or they're exponentially falling; they just don't do the trick. But now we have a way out: let the exponent be complex.

We are going to make a guess, called an *ansatz* in the business:

$$x(t) = Ae^{\alpha t} \tag{17.34}$$

where we will now allow α to be some general complex number.

The ansatz is a tentative guess with some parameters, A and α in this instance, the judicious choice of which may yield a solution. If you're lucky, it will work. If not, you move on and try another solution; it is just like speed dating.

So, let's take the ansatz in Eqn. 17.34, put it in Eqn. 17.32, and demand that it be satisfied:

$$\ddot{x} + \omega_0^2 x = 0 \tag{17.35}$$

$$\alpha^2 Ae^{\alpha t} + \omega_0^2 Ae^{\alpha t} = 0 \tag{17.36}$$

$$A(\alpha^2 + \omega_0^2)e^{\alpha t} = 0. \tag{17.37}$$

Our ansatz will work if we manage to get $(Ae^{\alpha t})(\alpha^2 + \omega_0^2)$ to vanish. How many ways are there to kill this beast? The choice $A = 0$ is called the trivial solution and corresponds to the oscillator sitting still forever. So A can be anything, except 0. Now $e^{\alpha t}$ never vanishes (even if α is complex), as here so it is not the cause of the zero. So it must be that

$$\alpha^2 + \omega_0^2 = 0 \quad \text{which means } \alpha = \pm i\omega_0. \tag{17.38}$$

(More generally, if in place of $e^{\alpha t}$, which *never vanishes*, we had any function that did not vanish identically, we can still cancel it by picking a time when it is non-zero, and deduce Eqn. 17.38.)

So now I have two solutions of the form $Ae^{\alpha t}$. For A you can pick any number you like, in fact, real or complex—it doesn't matter. The equation is satisfied. But α can be only one of two numbers: $\pm i\omega_0$.

How do we choose between the two solutions

$$x_+(t) = Ae^{i\omega_0 t} \quad \text{and} \tag{17.39}$$

$$x_-(t) = Ae^{-i\omega_0 t}? \tag{17.40}$$

It turns out that we can pick both, and I'll tell you what I mean by that. Let us begin with the fact that Eqn. 17.32 is a *homogeneous, linear* differential equation. I'll have to tell you what that means through an example:

$$17\frac{d^{96}x}{dx^{96}} + 16\frac{d^3x}{dx^3} + 2x = 0. \tag{17.41}$$

It is homogeneous because you only find a single power of x anywhere, which happens to be the first power here. It is a linear equation because you find either the function x or its derivatives, but never the squares of cubes or higher powers of x or the derivatives. Note that the 96-th derivative does not change this fact; it is still the 96-th derivative of x and not, say, x^3. By contrast,

$$\frac{d^2x}{dt^2} + 3x^2 = 0 \tag{17.42}$$

is a non-linear equation because of the x^2 term. A linear equation has a very important property that lies at the heart of so many things we do.

This is called the *principle of superposition*, and it states: *if $x_1(t)$ and $x_2(t)$ are two solutions of a homogeneous linear equation, then so is any linear combination of them with constant (t-independent) coefficients A and B*:

$$x(t) = Ax_1(t) + Bx_2(t).$$

Let us prove this for the oscillator case to understand where linearity comes in. Given

$$\ddot{x}_1 + \omega_0^2 x_1 = 0 \tag{17.43}$$
$$\ddot{x}_2 + \omega_0^2 x_2 = 0, \tag{17.44}$$

let us multiply the first by a constant A, the second by B, and add to get

$$A\ddot{x}_1 + A\omega_0^2 x_1 + B\ddot{x}_2 + B\omega_0^2 x_2 = 0 \tag{17.45}$$
$$\frac{d^2(Ax_1 + Bx_2)}{dt^2} + \omega_0^2(Ax_1 + Bx_2) = 0, \tag{17.46}$$

which clearly shows that $x(t) = Ax_1(t) + Bx_2(t)$ is also a solution. We used the fact that any derivative of a linear combination is the same linear combination of the derivatives and that the non-derivative term was linear in x. Try doing this for the non-linear case, say Eqn. 17.42, and you will find it does not work because $3Ax_1(t)^2 + 3Bx_2(t)^2 \neq 3(Ax_1(t) + Bx_2(t))^2$.

The bottom line is that if you give me two independent solutions to a homogeneous linear equation, I can manufacture an infinite number of solutions because I can pick A and B any way I like. The solutions x_1 and x_2 are like unit vectors $\mathbf{i}$ and $\mathbf{j}$, whose linear combinations with all possible coefficients yield an infinite number of vectors in two dimensions. A word of caution: $\mathbf{i}$ and $3\mathbf{i}$ are also two vectors, but by combining them you can only get solutions parallel to $\mathbf{i}$. These two vectors are said to be *linearly dependent*, which in this simple case means one is a multiple of the other. Likewise $e^{i\alpha t}$ and $5e^{i\alpha t}$ cannot be used to build anything other than multiples of $e^{i\alpha t}$. However, $e^{-i\alpha t}$ is an independent object because it is not a multiple of $e^{i\alpha t}$.

By the same analogy with **i** and **j**, if a linear combination of two linearly independent functions equals another linear combination, the coefficients have to match on both sides. Thus

$$Ae^{\alpha t} + Be^{5\alpha t} = Ce^{\alpha t} + De^{5\alpha t} \quad \text{implies} \tag{17.47}$$
$$A = C \qquad\qquad B = D. \tag{17.48}$$

17.3 Conditions on solutions to the harmonic oscillator

Let us then consider the general solution

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}. \tag{17.49}$$

How do we decide what A and B are? In general they are arbitrary. But on a given day, when you pull the mass by 9 cm and release it from rest, A and B have to be chosen so that at $t = 0$, $x(0) = 9$ and the velocity $v(0) = 0$. But I have a bigger problem. The answer is manifestly not real, and we know x is a real function. That is not a mathematical requirement of the equation, but a physical requirement. To say that x is real means the following. A complex number $x + iy$ has a complex conjugate $x - iy$, and the property of real numbers is that when you take the complex conjugate, nothing happens: it satisfies the condition $z = z^*$. There is no imaginary part whose sign you can flip. Real numbers are their own complex conjugates.

So, I'm going to demand that this solution, in addition to satisfying the basic equation, also is real. To do that, I'm going to demand $x(t)$ equals its complex conjugate $x^*(t)$:

$$x^*(t) = A^* e^{-i\omega_0 t} + B^* e^{+i\omega_0 t} = x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t}. \tag{17.50}$$

To find $x^*(t)$ given $x(t)$, I conjugated everything in sight. The complex conjugates of A and B became A^* and B^*. The complex conjugate of $e^{+i\omega_0 t}$ is $e^{-i\omega_0 t}$ and vice versa, because the i goes to minus i while t and ω_0 are real numbers and nothing happens to them.

So $x(t) = x^*(t)$ for all times t, if the coefficients of $e^{\pm i\omega_0 t}$ in Eqn. 17.50 match:

$$A = B^* \qquad B = A^*. \tag{17.51}$$

However, if $A = B^*$, then $B = A^*$ follows automatically because both are saying the same thing: A and B have equal real parts and opposite imaginary parts. This can also be seen another way. Conjugating both sides of $A = B^*$, we get $A^* = (B^*)^* = B$ because conjugating any complex number twice changes the sign of its imaginary part twice, which is equivalent to doing nothing: $(z^*)^* = z$.

The reality of x then leads to the solution

$$x(t) = Ae^{i\omega_0 t} + A^* e^{-i\omega_0 t}. \tag{17.52}$$

In other words, B is not an independent number; it has to be the complex conjugate of A if x is to be real. I hope you can see at a glance that the solution above is real, because whatever the first animal is, the second is its complex conjugate and has the opposite imaginary part. When you add them, the answer will be real. But A is not necessarily real. In polar form it has a modulus $|A|$ and a phase ϕ, so that

$$\begin{aligned} x(t) =& |A|e^{i\phi} e^{i\omega_0 t} + |A|e^{-i\phi} e^{-i\omega_0 t} = |A|e^{i(\phi+\omega_0 t)} + |A|e^{-i(\phi+\omega_0 t)} \\ &= |A|\left[e^{i(\phi+\omega_0 t)} + e^{-i(\phi+\omega_0 t)}\right]. \end{aligned} \tag{17.53}$$

Now, what is this function I have in brackets? You should be able to recognize this creature as a cosine. We have ended up with

$$x(t) = 2|A|\cos(\omega_0 t + \phi). \tag{17.54}$$

This describes an oscillator of amplitude $2|A|$ and phase ϕ. Notice how the amplitude and phase of the oscillator were encoded in a *single* complex number A.

Suppose you had chosen to use $\sin\omega_0 t$ and $\cos\omega_0 t$ as the two basic solutions instead of $e^{\pm i\omega_0 t}$. The general solution would have been

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t \tag{17.55}$$

where A and B are arbitrary. However, demanding that x be real will force them both to be real. No matter how you slice it, a physical oscillator will have in its solution just two free parameters: they could be two real numbers A and B as above or one complex number $A = |A|e^{i\phi}$ as before.

Now, this is a long and difficult way to get back the old answer. Your reaction may be, “We don’t need these complex numbers. We have enough problems in life; we’re doing well with sines and cosines, thank you.” But now I’m going to give you a problem where you cannot talk your way out by just turning it into a word problem.

17.4 Exponential functions as generic solutions

Here is the problem: a mass m, connected to a spring of force constant k, is moving on a surface with friction. The minute there is friction, you have an extra force. We know that if you’re moving to the right, the force of friction is to the left, and, if you are moving to the left, the force is to the right, that is, the frictional force is velocity dependent. The equation that crudely models this velocity dependence is

$$m\ddot{x} = -kx - \gamma\, m\dot{x} \tag{17.56}$$

where I include a factor m in the frictional coefficient to simplify subsequent algebra. Dividing by m, our equation becomes

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2 x = 0. \tag{17.57}$$

Can you solve this as a word problem? It’s going to be difficult, because you want a function that, when you take two derivatives, add some amount of its own derivative, and then some of itself, gives zero. It is not clear a trigonometric function can do that. However, an exponential has to work because it reproduces itself no matter how many derivatives you take. Thus we make the ansatz

$$x(t) = Ae^{\alpha t}. \tag{17.58}$$

Note that I do not explicitly use a complex exponential. If α is meant to be complex, it will come out that way; we are not forcing it to be real in making this ansatz. When we feed it into Eqn. 17.57 we find, because every derivative brings a factor of α,

$$A(\alpha^2 + \gamma\,\alpha + \omega_0^2)e^{\alpha t} = 0. \tag{17.59}$$

Once again, A cannot be the cause of the zero, because if A vanishes you've killed the whole solution and $e^{\alpha t}$ is not going to vanish, so the only way is for the stuff in brackets to vanish:

$$(\alpha^2 + \gamma\alpha + \omega_0^2) = 0. \tag{17.60}$$

That means the α that you put into this guess must be one of the roots

$$\alpha_\pm = \frac{-\gamma \pm \sqrt{\gamma^2 - 4\omega_0^2}}{2} = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2}. \tag{17.61}$$

The general solution is

$$x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{17.62}$$

$$= A\exp\left[\left(-\frac{\gamma}{2} + \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right] + B\exp\left[\left(-\frac{\gamma}{2} - \sqrt{\frac{\gamma^2}{4} - \omega_0^2}\right)t\right]. \tag{17.63}$$

The motion described by the solution depends on the value of $\frac{\gamma}{2\omega_0}$.

17.5 Damped oscillations: a classification

Let us classify the different kinds of behavior that emerge as we vary $\frac{\gamma}{2\omega_0}$.

17.5.1 Over-damped oscillations

We first consider the *over-damped case*

$$\frac{\gamma}{2} > \omega_0. \tag{17.64}$$

In this case both roots $\alpha_\pm$ are real and both are negative: α_- is negative being a sum of two negative numbers, while α_+ is negative because the positive square root is smaller than $\gamma/2$. This means that $x(t \to \infty) \to 0$, which is in accord with our expectation that friction will eventually bring the oscillations to an end.

How about A and B? First of all, they are both real as can be seen by equating $x(t)$ to its conjugate. Because the exponentials are real they do not respond to conjugation and we require $A = A^*$ and $B = B^*$.

To find A and B, we need two pieces of data, which I will take to be initial position, $x(0)$, and the initial velocity, $v(0)$. If we put $t = 0$ in Eqn. 17.62 we find

$$x(0) = A + B. \tag{17.65}$$

Next I take the derivative of Eqn. 17.63 *and then set* $t = 0$ to find

$$v(0) = A\alpha_+ + B\alpha_-. \tag{17.66}$$

Solving these simultaneous equations will yield A and B. To test yourself, try showing that if the oscillator is displaced to some $x(0) > 0$ and released from rest, that is, $v(0) = 0$, then $x(t)$ never becomes 0 and hence cannot become negative. This means the mass will simply relax to its equilibrium position without any oscillations.

17.5.2 Under-damped oscillations

In turning on friction we got carried away: from being 0 in the very first example, γ jumped to a value greater than $2\omega_0$. Consider now the intermediate case when $0 < \gamma < 2\omega_0$. What do the solutions look like now? We should be able to guess that, at least for very tiny values of γ, the oscillator will oscillate as before, but with a slowly diminishing amplitude. Let us verify and quantify this expectation.

The roots now become

$$\alpha_\pm = -\frac{\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} - \omega_0^2} \tag{17.67}$$

$$= -\frac{\gamma}{2} \pm i\sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \tag{17.68}$$

$$\equiv -\frac{\gamma}{2} \pm i\omega'. \tag{17.69}$$

We have introduced yet another frequency

$$\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} < \omega_0, \tag{17.70}$$

which describes the oscillatory part of the motion. Note that the roots are complex conjugates

$$\alpha_+ = \alpha_-^* \tag{17.71}$$

and the general solution becomes

$$x(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{17.72}$$

$$= e^{-\frac{1}{2}\gamma t}\left[Ae^{i\omega' t} + Be^{-i\omega' t}\right]. \tag{17.73}$$

I leave it to you to verify that once again $x = x^*$ implies $A^* = B$ because the A and B terms get exchanged under complex conjugation. Repeating the analysis for the case $\gamma = 0$, this solution may be rewritten as

$$x(t) = Ce^{-\frac{1}{2}\gamma t}\cos\left[\omega' t + \phi\right] \quad \text{where} \tag{17.74}$$

$$C = 2|A| \quad \text{and} \quad A = |A|e^{i\phi}. \tag{17.75}$$

Figure 17.4 shows what the damped oscillation looks like for $A = 2$, $\gamma = 1$, and $\omega' = 2\pi$. This is typically what you will see if you excite any

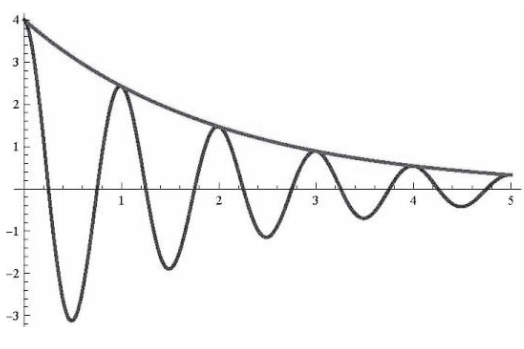

Figure 17.4 Damped oscillations with $x(t) = 4e^{-.5t}\cos(2\pi t)$, i.e., $A = 2$, $\gamma = 1$, and $\omega' = 2\pi$. The falling exponential shows the decay of the amplitude.

system with some modest amount of frictional loss. If γ is very small, you may not realize the oscillations are being damped.

17.5.3 Critically damped oscillations

Having considered the cases $\gamma > 2\omega_0$ (over-damped) and $\gamma < 2\omega_0$ (under-damped), we turn to the *critically damped* case $\gamma = 2\omega_0$. In this case $\alpha_+ = \alpha_- = -\frac{\gamma}{2}$. Where is the second solution to accompany $Ae^{-\frac{\gamma t}{2}}$? We know in every problem there must be two solutions, because we should be able to pick the initial position and velocity at will. That's an area of mathematics I don't want to enter now, but you can verify that the second solution is $Bte^{-\frac{\gamma t}{2}}$, which is not a pure exponential. You will find the derivation of this solution in my math book. The general solution for the critically damped case is thus

$$x(t) = e^{-\frac{\gamma t}{2}}\left[A + Bt\right]. \tag{17.76}$$

Try to show in this case that $A = x(0)$ and $B = v(0) + \frac{\gamma}{2}x(0)$.

17.6 Driven oscillator

Next we turn to a more challenging problem. I have, as before, the mass, the spring, and friction. But now I'm going to apply an extra force, $F_0 \cos\omega t$. This is called a *driven oscillator.* Imagine that I am actively shaking the mass with my hand, exerting the force $F_0 \cos\omega t$. Now there are three ω's: $\omega_0 = \sqrt{\frac{k}{m}}$, the natural frequency of the undamped free oscillator; $\omega' = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$, which entered the under-damped oscillator; and finally ω, the frequency of the driving force, which is completely up to me to choose. The equation to solve is

$$m\ddot{x} + \gamma\, m\dot{x} + kx = F_0 \cos\omega t, \tag{17.77}$$

which we rewrite as

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t. \tag{17.78}$$

This problem is difficult because you cannot guess the answer to it by turning it into a word problem: neither $x(t) \propto \cos(\omega t)$ nor $x(t) \propto \sin(\omega t)$ is a good ansatz because you cannot have all four terms be functions of the same kind as the ansatz. In fact, only an exponential can lead to all four terms being the same functional form (exponential) because taking any number of derivatives will leave it alone. *But our driving force is a cosine and not an exponential.*

Here is a clever trick to beat this problem. Recall that with no driving force, if

$$\ddot{x}_1 + \gamma\,\dot{x}_1 + \omega_0^2 x_1 = 0 \quad \text{and} \tag{17.79}$$
$$\ddot{x}_2 + \gamma\,\dot{x}_2 + \omega_0^2 x_2 = 0, \tag{17.80}$$

then multiplying the first by a constant A and the second by a constant B and adding, we found that $Ax_1 + Bx_2$ was also a solution:

$$\frac{d^2[Ax_1 + Bx_2]}{dt^2} + \gamma\,\frac{d[Ax_1 + Bx_2]}{dt} + \omega_0^2\,[Ax_1 + Bx_2] = 0. \tag{17.81}$$

I have used the fact that the derivatives of a linear combination $Ax_1 + Bx_2$ is the same linear combination of the derivatives.

Suppose now that there is a driving force behind x_1 and x_2:

$$\ddot{x}_1 + \gamma\,\dot{x}_1 + \omega_0^2 x_1 = \frac{F_1(t)}{m} \tag{17.82}$$
$$\ddot{x}_2 + \gamma\,\dot{x}_2 + \omega_0^2 x_2 = \frac{F_2(t)}{m}. \tag{17.83}$$

It follows by the same manipulations that

$$\frac{d^2[Ax_1 + Bx_2]}{dt^2} + \gamma\,\frac{d[Ax_1 + Bx_2]}{dt} + \omega_0^2\,[Ax_1 + Bx_2] = A\frac{F_1(t)}{m} + B\frac{F_2(t)}{m}. \tag{17.84}$$

In other words, *in a linear equation, the response to a linear combination of forces is the corresponding linear combination of responses.*

Now for the trick. Let $x(t)$ be the solution to

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{17.85}$$

and $y(t)$ the solution to

$$\ddot{y} + \gamma\,\dot{y} + \omega_0^2 y = \frac{F_0}{m}\sin\omega t. \tag{17.86}$$

(We could call these two solutions x_1 and x_2, but there is a reason for this nomenclature.) Multiply the second equation by i and add it to the first to obtain

$$\frac{d^2[x+iy]}{dt^2} + \gamma\,\frac{d[x+iy]}{dt} + \omega_0^2[x+iy] = \frac{F_0}{m}(\cos\omega t + i\sin\omega t)$$
$$= \frac{F_0}{m}e^{i\omega t} \tag{17.87}$$

$$\ddot{z} + \gamma\,\dot{z} + \omega_0^2 z = \frac{F_0}{m}e^{i\omega t} \quad \text{where} \tag{17.88}$$
$$z(t) = x(t) + iy(t). \tag{17.89}$$

This is a special case of Eqn. 17.84 with $A = 1$ and $B = i$.

So, in Eqn. 17.88 I have manufactured a problem in which the thing that's vibrating is not a real number, but $z = x + iy$. The force driving it is also not a real number; it is $\frac{F_0}{m}e^{i\omega t}$. *The point is that if I can solve the problem somehow, I can get $x(t)$ as the real part of the answer.* (The imaginary part of it, $y(t)$, will be the solution to the fictitious Eqn. 17.86 I concocted.)

And I can solve Eqn. 17.88 for $z(t)$ very easily because I can now make the ansatz

$$z(t) = z_0 e^{i\omega t}. \tag{17.90}$$

Because every derivative pulls out an $i\omega$ we have

$$\left[-\omega^2 + i\omega\gamma + \omega_0^2\right] z_0 e^{i\omega t} = \frac{F_0}{m}e^{i\omega t}. \tag{17.91}$$

We may safely cancel $e^{i\omega t}$ because it is not identically zero to obtain the equation for z_0:

$$z_0 = \frac{F_0/m}{\left[-\omega^2 + i\omega\gamma + \omega_0^2\right]} \tag{17.92}$$

$$= \frac{F_0/m}{Z(\omega)} \quad \text{where we have defined} \tag{17.93}$$

$$Z(\omega) = \left[-\omega^2 + i\omega\gamma + \omega_0^2\right]. \tag{17.94}$$

The magic of the exponential is that the differential equation 17.88 has reduced to an algebraic equation for the (complex) amplitude z_0

$$Z(\omega)z_0 = \frac{F_0}{m}, \tag{17.95}$$

which is solved by dividing both sides by $Z(\omega)$:

$$z_0 = \frac{F_0/m}{Z(\omega)}. \tag{17.96}$$

It follows that

$$z(t) = z_0 e^{i\omega t} = \frac{[F_0/m]\, e^{i\omega t}}{Z(\omega)}. \tag{17.97}$$

All we need to do now is take the real part to get $x(t)$. If you thought that this means replacing $e^{i\omega t}$ by $\cos\omega t$ you are wrong, because

$$Z(\omega) = \left[-\omega^2 + i\omega\gamma + \omega_0^2\right] \tag{17.98}$$

is itself a complex number whose real and imaginary parts can mix with the real and imaginary part of $e^{i\omega t}$. So here is the correct way to do this. Take $Z(\omega)$ in Cartesian form

$$Z(\omega) = \left[\omega_0^2 - \omega^2\right] + i\omega\gamma \tag{17.99}$$

and write it in polar form

$$Z(\omega) = |Z|e^{i\phi} \quad \text{where} \tag{17.100}$$

$$|Z| = \sqrt{\left[\omega_0^2 - \omega^2\right]^2 + \omega^2\gamma^2} \tag{17.101}$$

$$\phi = \tan^{-1}\left[\frac{\omega\gamma}{\omega_0^2 - \omega^2}\right]. \tag{17.102}$$

Figure 17.5 shows Z in the complex plane.

Return to Eqn. 17.97 with this result to obtain

$$z(t) = \frac{[F_0/m]e^{i\omega t}}{Z(\omega)} = \frac{[F_0/m]\,e^{i\omega t}}{|Z|e^{i\phi}} \tag{17.103}$$

$$= \frac{F_0}{m|Z|}e^{i(\omega t - \phi)}. \tag{17.104}$$

Now we can take the real part easily because $\frac{F_0}{m|Z|}$ is real. Here is the final answer:

$$x(t) = \frac{F_0}{m|Z|}\cos(\omega t - \phi) \equiv x_0 \cos(\omega t - \phi). \tag{17.105}$$

Notice that the cause, $\frac{F_0}{m}\cos(\omega t)$, produces an effect that is reduced in magnitude by $|Z|$ and shifted in phase into $\cos(\omega t - \phi)$. While there is a way to obtain both these transformations with real numbers, it is so much

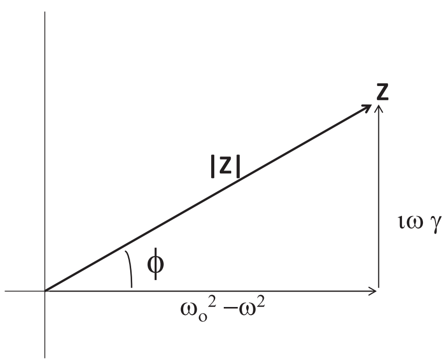

Figure 17.5 The complex number $Z(\omega)$ in its Cartesian and polar forms.

easier with complex numbers: dividing the force by a complex number $Z = |Z|e^{i\phi}$ achieves both these effects in one shot. Bear in mind that the phase ϕ cannot be eliminated by choice of the origin in time because it is the phase *relative* to that of the applied force $F_0 \cos \omega t$.

Let us pause to analyze Eqn. 17.105. Keeping $\frac{F_0}{m}$ fixed, let us vary ω, the frequency of the driving force, to see what happens to x_0, the amplitude of vibrations. When $\omega = 0$, that is, when the force does not vary with time, we find

$$|Z(0)| = \sqrt{\left[\omega_0^2 - \omega^2\right]^2 + \omega^2\gamma^2}\bigg|_{\omega=0} = \omega_0^2 \tag{17.106}$$

so that

$$x_0 = \frac{F_0}{m\omega_0^2} = \frac{F}{k}, \tag{17.107}$$

which makes sense: a constant force F will produce a displacement $\frac{F}{k}$.

When $\omega \to \infty$, we find $x_0 \to 0$. Somewhere in between these extremes, the response peaks. It is clear that if γ is very small, we get the biggest response when $\omega = \omega_0$: this is when $|Z|$ is the smallest. This is called *resonance*. It tells us that the response of the system to a driving force is greatest when the driving frequency equals the natural frequency. Imagine you are pushing a kid on a swing, by periodically supplying the force. If you are not paying attention and pushing at your own frequency, sometimes you will slow the kid and sometimes you will speed up the kid. It is best to push exactly when the kid is moving away from you. Note that in a real swing $\gamma > 0$, and there is no danger of the kid flying off to infinity.

Radios exploit the phenomenon of resonance. Right now this room is filled with electromagnetic signals from many stations, and yet you are able to listen to the one you want. The trick is that you can adjust the natural frequency of the electrical circuits picking up the signal by turning the dial to match that of the station of interest. For this plan to succeed, you need the graph in Figure 17.6 to be extremely sharp. Imagine that there are just two stations at two frequencies. Even if you tune the radio to resonate with one, you will be getting a tiny response from the tail of the other one. The goal is to keep this interference to a minimum.

Where are the free parameters in this problem? Everything seems determined in Eqn. 17.105. What if this solution does not agree with some

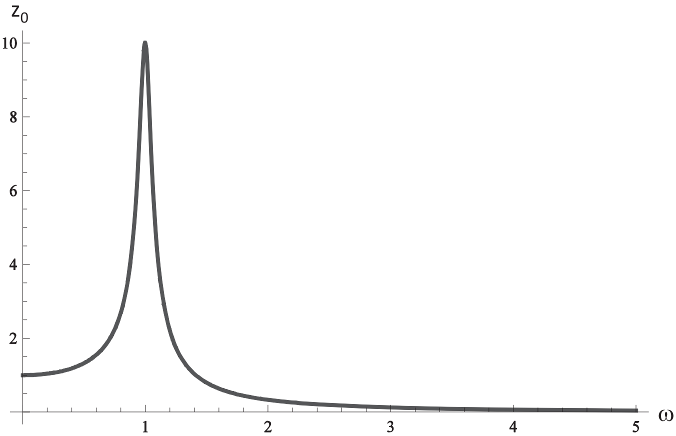

Figure 17.6 The amplitude $z_0(\omega)$ for a system with $\omega_0 = 1$ and $\gamma = 0.1$ driven by an external force with $\frac{F_0}{m} = 1$.

initial condition, such as a specified $x(0)$ or $v(0)$? The answer is that we can add to this solution (called the *particular solution*) any solution to the equation with $F(t) = 0$, referred to as the *complementary solution* and given in Eqn. 17.73. Thus the most general solution to the driven oscillator is

$$x(t) = \frac{F_0}{m|Z|}\cos(\omega t - \phi) + e^{-\frac{1}{2}\gamma t}\left[Ae^{i\omega' t} + Be^{-i\omega' t}\right]. \quad (17.108)$$

Even after adding this term $x(t)$ satisfies Eqn. 17.78 because the added terms disappear when we compute the left-hand side. Another way to see this is to invoke superposition: consider the right-hand side of Eqn. 17.78 to be $\frac{F_0}{m}\cos(\omega t) + 0$ and add the response due to 0, which is the complementary function. The numbers A and B can once again be chosen to match the initial conditions, say the initial position and velocity. One may forget about the complementary function at large times, because it dies out exponentially.

Finally, consider the force pushing the kid as described earlier. It is periodic but not simply the function $\cos\omega t$. (For example, the force on the kid acts only for a small part of each period, while the cosine is non-zero except twice in a period.) Amazingly, we can use the technique described above to find the response to any periodic force, not necessarily a simple cosine or oscillatory exponential function. This is thanks to the mathematician Joseph Fourier, who showed that any function $F(t)$ with period T

may be written as a sum of oscillating exponentials, with suitable periods, multiplied by suitable coefficients F_n:

$$F(t) = \sum_{n=-\infty}^{\infty} F_n e^{2\pi\, int/T} \equiv \sum_{n=-\infty}^{\infty} F_n e^{i\omega_n t} \text{ where} \tag{17.109}$$

$$\omega_n = \frac{2\pi\, n}{T}. \tag{17.110}$$

In the right-hand sides of Eqns. 17.109 and 17.110, we have a sum of forces with frequencies $\omega_n = \frac{2\pi\, n}{T}$. I state without proof that the coefficients are determined by the given $F(t)$ as follows:

$$F_n = \frac{1}{T}\int_0^T F(t)e^{-i\omega_n t}dt. \tag{17.111}$$

We are done because we know the response $z_0(n)$ due to each oscillating term $F_n e^{i\omega_n t}$ in the sum, and by previous linearity arguments the total response is the corresponding sum over responses:

$$z(t) = \sum_{n=-\infty}^{\infty} z_0(n)e^{i\omega_n t} \text{ where} \tag{17.112}$$

$$z_0(n) = \frac{\frac{F_n}{m}}{Z(\omega_n)}. \tag{17.113}$$

If the driving force $F(t)$ is real, the $z(t)$ above will automatically turn out to be real. If you want more practice using complex numbers, you are invited to read the following proof.

First note that

$$-\omega_n = \frac{2\pi\,(-n)}{T} = \omega_{(-n)} \tag{17.114}$$

$$F_n^* = \frac{1}{T}\int_0^T F(t)e^{+i\omega_n t}dt \tag{17.115}$$

$$= \frac{1}{T}\int_0^T F(t)e^{-i(-\omega_n)t}dt = \frac{1}{T}\int_0^T F(t)e^{-i\omega_{(-n)}t}dt \tag{17.116}$$

$$= F_{-n}. \tag{17.117}$$

Now pair the contributions from the terms in Eqn. 17.112 with any n and $-n$:

$$z_0(n)e^{i\omega_n t} + z_0(-n)e^{i\omega_{(-n)} t} = \frac{\frac{F_n}{m}}{Z(\omega_n)}e^{i\omega_n t} + \frac{\frac{F_{-n}}{m}}{Z(\omega_{-n})}e^{i\omega_{(-n)} t} \tag{17.118}$$

$$= \frac{\frac{F_n}{m}}{Z(\omega_n)}e^{i\omega_n t} + \left[\frac{\frac{F_n}{m}}{Z(\omega_n)}e^{i\omega_n t}\right]^*, \tag{17.119}$$

which is manifestly real, being a sum of something plus its conjugate. We have also used

$$Z(\omega_{(-n)}) = Z(-\omega_n) = Z^*(\omega_n), \tag{17.120}$$

because for any ω we have

$$Z(\omega) = \left[-\omega^2 + i\omega\gamma + \omega_0^2\right] \tag{17.121}$$

$$Z^*(\omega) = \left[-\omega^2 - i\omega\gamma + \omega_0^2\right] = Z(-\omega). \tag{17.122}$$