CHAPTER 6

Conservation of Energy in $d = 2$

6.1 Calculus review

We begin with some mathematical preparation for what I'm going to do next. Let's take some function $f(x)$ shown in Figure 6.1. I start at some point x with a value $f(x)$. When I go to a nearby point, $x + \Delta x$, the function changes by $\Delta f = f(x + \Delta x) - f(x)$. All these tiny quantities are exaggerated in the figure so you can see them. We are going to need approximations to the change in the function as $\Delta x \to 0$. A common one is to pretend the function is linear with the local value of the slope $f'(x) = \frac{df}{dx}$, as depicted

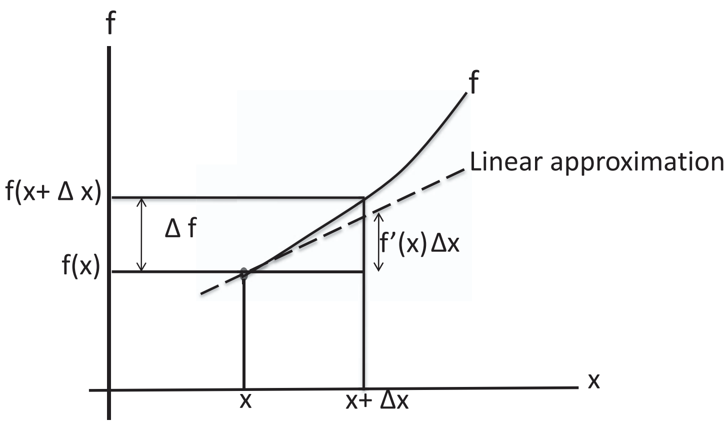

Figure 6.1 The change, Δf, in a function $f(x)$ as x changes by Δx, may be approximated by $\Delta f \simeq f'(x)\Delta x$, where $f'(x) = \frac{df}{dx}$. The solid line is the actual function and the dotted line is the approximation by a straight line of slope $f'(x)$.

by the dotted line. The change in f along the straight line is $f'(x)\Delta x$. It differs from the actual Δf by a tiny amount because the function is not following the same slope that you have to begin with; it's curving up. Take a concrete example:

$$f(x) = x^2 \tag{6.1}$$

$$f(x+\Delta x) = x^2 + 2x\Delta x + (\Delta x)^2 \tag{6.2}$$

$$\Delta f = 2x\Delta x + (\Delta x)^2 \tag{6.3}$$

$$\Delta f = f'(x)\Delta x + (\Delta x)^2. \tag{6.4}$$

This result is valid for Δx of any size. We see that the exact change is $f'(x)\Delta x$ plus something quadratic in Δx. If we are interested in very small Δx, we may start ignoring all but the term linear in Δx:

$$\Delta f = f'(x)\Delta x + \mathcal{O}(\Delta x)^2 \tag{6.5}$$

where $\mathcal{O}(\Delta x)^2$ signifies that the neglected terms are of order $(\Delta x)^2$ and higher.

Often we will use

$$\Delta f \simeq f'(x)\Delta x \tag{6.6}$$

as an approximation for small Δx.

Consider, for example, $f(x) = (1+x)^n$ and its values near $x = 0$. Clearly $f(0) = 1$. Suppose you want the function at a point x very close to the origin. In this case $\Delta x = x - 0$ is just x itself and the approximate value will be

$$f(x) = f(0) + f'(0)x + \ldots = 1 + n\,(1+x)^{n-1}\big|_{x=0}\, x + \ldots = 1 + nx + \ldots, \tag{6.7}$$

a result we will exploit mercilessly.

On other occasions, we will take the limit $\Delta x \to 0$ in the end and write the equality

$$df = f'(x)dx \tag{6.8}$$

with the understanding that both sides are to be integrated to obtain

$$\int_1^2 df = f(x_2) - f(x_1) = \int_{x_1}^{x_2} f'(x)dx. \tag{6.9}$$

6.2 Work done in $d = 2$

Now we are going to derive the work-energy theorem and the law of conservation of energy in two dimensions. I am hoping I will get some relation like $K_1 + U_1 = K_2 + U_2$, where $U = U(x, y)$. How do you visualize the function of two variables $f(x, y)$? On top of each point (x, y) you measure the value of $f(x, y)$ in the third perpendicular direction. The function defines a surface over the $x - y$ plane and the distance from the plane to the surface is the value of f at the point (x, y). For example, (x, y) could be coordinates of a point in the United States and the function could be the temperature $T(x, y)$ at that point. So you plot on top of each point in the United States the local temperature.

Once I have got the notion of a function of two variables, I want to move around the plane and ask how the function changes. But now I have an infinite number of options. I can move along x, I can move along y, I can move at some intermediate angle. Consider derivatives along the two principal directions x and y. We're going to define a *partial derivative* as follows. You start at the point (x, y), go to the point $(x + \Delta x, y)$, subtract the function at the starting point, divide by Δx, and take $\Delta x \to 0$. This defines the *partial derivative with respect to x*:

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}. \tag{6.10}$$

The curly ∂ instead of d tells you it's the partial derivative. As you move horizontally, you notice you don't do anything to y. We could make it very explicit by using a subscript y as follows:

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}. \tag{6.11}$$

We will not do that: if one coordinate is being varied, all the others (of which there is just one in $d=2$) will be assumed fixed. In the same notation

$$\left.\frac{\partial f}{\partial y}\right|_x \equiv \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}. \tag{6.12}$$

Let's get some practice with $f = x^3 y^2$. To find $\frac{\partial f}{\partial x}$ we see how f varies with x keeping y constant. That means we treat y like a number such as 5 when we encounter it. So we have

$$\left.\frac{\partial f}{\partial x}\right|_y = 3x^2 y^2 \tag{6.13}$$

$$\left.\frac{\partial f}{\partial y}\right|_x = 2x^3 y. \tag{6.14}$$

We know from the calculus of one variable that you can take the derivative of the derivative. Here are the four possible second derivatives and their explicit values for $f = x^3 y^2$:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial x^2} = 6xy^2 \tag{6.15}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial y^2} = 2x^3 \tag{6.16}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \equiv \frac{\partial^2 f}{\partial x \partial y} = 6x^2 y \tag{6.17}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \equiv \frac{\partial^2 f}{\partial y \partial x} = 6x^2 y. \tag{6.18}$$

Notice that the *mixed or cross derivatives* are equal:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}. \tag{6.19}$$

That's a property of the generic functions that we will encounter. I'd like to give you a feeling for why that is true. For what follows, bear in mind that when you make a small displacement in the plane, the change in any function is approximately

$$\Delta f \simeq \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y, \tag{6.20}$$

which becomes an equality in the limit $\Delta x \to 0$, $\Delta y \to 0$, and $\Delta f \to 0$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy. \tag{6.21}$$

These limits appear naturally when we plan to sum over the infinitesimal changes to get the corresponding integrals.

Armed with this, let us ask how much the function changes when we go from some point (x, y) to $(x + dx, y + dy)$ in Figure 6.2. We're going to

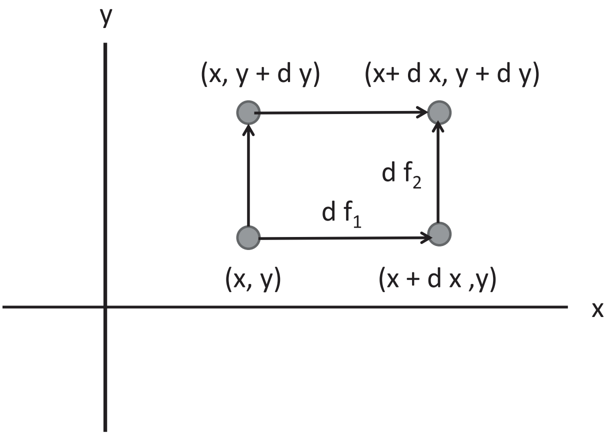

Figure 6.2 Two ways to go from (x, y) to $(x + dx, y + dy)$: move horizontally and then vertically or vice versa. That the change in f must be the same both ways becomes the requirement that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$.

make the move in two stages. We go via an intermediate point $(x + dx, y)$ and add the changes df_1 and df_2 in each step:

$$df_1 = \left.\frac{\partial f}{\partial x}\right|_{(x,y)} dx \tag{6.22}$$

$$df_2 = \left.\frac{\partial f}{\partial y}\right|_{(x+dx,y)} dy \tag{6.23}$$

$$df = \left.\frac{\partial f}{\partial x}\right|_{(x,y)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x+dx,y)} dy. \tag{6.24}$$

Notice that the second step requires the y partial derivative at $(x + dx, y)$. Because the partial derivative is itself just another function of x and y, we may write to leading order in dx

$$\left.\frac{\partial f}{\partial y}\right|_{(x+dx,y)} = \left.\frac{\partial f}{\partial y}\right|_{(x,y)} + \frac{\partial^2 f}{\partial x \partial y} dx. \tag{6.25}$$

Upon feeding this into Eqn. 6.24 we find

$$df = \left.\frac{\partial f}{\partial x}\right|_{(x,y)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x,y)} dy + \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)} dxdy. \tag{6.26}$$

If we first moved up to $(x, y + dy)$ and then to $(x + dx, y + dy)$, we would get a change in f with x and y interchanged. Equating the results from the two ways to find the change in f between (x, y) and $(x + dx, y + dy)$ we find

$$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(x,y)} dxdy = \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(x,y)} dydx. \tag{6.27}$$

Canceling the products of the infinitesimals, we get the equality of the mixed derivatives.

6.3 Work done in $d = 2$ and the dot product

Let us come back to deriving the law of conservation of energy in two dimensions. In one dimension we found that if $K = \frac{1}{2}mv^2$,

$$\frac{dK}{dt} = mv\frac{dv}{dt} = mva = Fv = F\frac{dx}{dt} \quad (6.28)$$

$$dK = Fdx \quad \text{upon canceling } dt \text{ above} \quad (6.29)$$

$$K_2 - K_1 = \int_{x_1}^{x_2} F(x)dx \quad \text{upon integrating both sides above} \quad (6.30)$$

$$= U(x_1) - U(x_2), \quad \text{which can be rearranged to give} \quad (6.31)$$

$$K_2 + U_2 = K_1 + U_1 \quad (6.32)$$

provided F did not depend on anything else besides x, such as $v(x)$.

We want to try the same thing in two dimensions. What expression should I use for the work done in two dimensions, given that the force and displacement are both vectors with two components each? How should I multiply all these parts in generalizing $dW = Fdx$? Here is the solution. I'm going to find $\frac{dK}{dt}$ for a body moving in two dimensions and call that the power $P = \frac{dW}{dt}$ just as in $d = 1$. For that I need a formula for kinetic energy. The obvious choice that reduces to what we know is correct for motion along just x or y is

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(v_x^2 + v_y^2). \quad (6.33)$$

Now we find

$$\frac{dK}{dt} = m\left(v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt}\right) \quad (6.34)$$

$$= F_x v_x + F_y v_y = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} \quad (6.35)$$

$$dK = F_x dx + F_y dy \quad (6.36)$$

where I have used Newton's second law $\mathbf{F} = m\frac{d\mathbf{v}}{dt}$ and multiplied both sides of Eqn. 6.35 by dt, which is allowed in the sense explained earlier. If I define the work done as

$$dW = F_x dx + F_y dy, \tag{6.37}$$

I find, just as in $d = 1$, that

$$dK = dW = F_x dx + F_y dy. \tag{6.38}$$

The force and displacement are both vectors

$$\mathbf{F} = \mathbf{i}F_x + \mathbf{j}F_y \tag{6.39}$$
$$d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy \tag{6.40}$$

and their components enter dW in the combination $dW = F_x dx + F_y dy$. Likewise the power P may be written as

$$P = \frac{dK}{dt} = F_x v_x + F_y v_y. \tag{6.41}$$

Given two vectors

$$\mathbf{A} = \mathbf{i}A_x + \mathbf{j}A_y \tag{6.42}$$
$$\mathbf{B} = \mathbf{i}B_x + \mathbf{j}B_y, \tag{6.43}$$

we see that the combination $A_x B_x + A_y B_y$ appears very naturally. It has a name: *the dot product of* **A** *and* **B**, denoted by $\mathbf{A} \cdot \mathbf{B}$. That is, by definition,

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y. \tag{6.44}$$

In this notation

$$dW = \mathbf{F} \cdot d\mathbf{r} \tag{6.45}$$
$$P = \mathbf{F} \cdot \mathbf{v}. \tag{6.46}$$

A few factoids about $\mathbf{A} \cdot \mathbf{B}$. First

$$\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 = A^2 \tag{6.47}$$

where A is the length of **A**.

Next if θ_A and θ_B are the angles **A** and **B** make with the x-axis, then

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y \tag{6.48}$$
$$= A\cos\theta_A B\cos\theta_B + A\sin\theta_A B\sin\theta_B \tag{6.49}$$
$$= AB[\cos\theta_A \ \cos\theta_B + \sin\theta_A \ \sin\theta_B] \tag{6.50}$$
$$= AB\cos[\theta_B - \theta_A] = AB\cos[\theta_A - \theta_B], \tag{6.51}$$

which is usually written more compactly as

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \tag{6.52}$$

where it is understood θ is the angle between the vectors. It can be measured from **A** to **B** or the other way since $\cos\theta$ is unaffected by a sign change in θ.

Equation 6.52 works even in $d = 3$ because **A** and **B** can still be made to lie in a plane and θ defined as the angle between them in this plane. However, in terms of components we must bring in all three components:

$$\mathbf{A}\cdot\mathbf{B} = A_x B_x + A_y B_y + A_z B_z, \tag{6.53}$$

a result that seems reasonable and one which can be verified after some messy trigonometry.

The dot product is symmetric since $\cos\theta = \cos(-\theta)$:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}. \tag{6.54}$$

Note that if we set $\mathbf{A} = \mathbf{B}$, then $\mathbf{A}\cdot\mathbf{A} = AA\cos 0 = A^2$.

The two definitions of the dot product, Eqns. 6.44 and 6.52, are fully equivalent. If you are thinking in terms of the components, $A_x B_x + A_y B_y$ is more natural, while if you are thinking in terms of arrows of some lengths and angles, $AB\cos\theta$ is preferred. Which one you use depends on your goals.

For example, to establish an important property that the dot product is distributive:

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \tag{6.55}$$

it is easier to proceed as follows

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = A_x(B_x+C_x)+A_y(B_y+C_y) \tag{6.56}$$
$$= A_xB_x + A_yB_y + A_xC_x + A_yC_y = \mathbf{A}\cdot\mathbf{B}+\mathbf{A}\cdot\mathbf{C}. \tag{6.57}$$

On the other hand, using $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, it is easier to establish the following very useful results:

- If **A** and **B** are parallel, i.e., $\theta = 0$, their dot product is a maximum.
- If **A** and **B** are perpendicular, their dot product is zero.
- Under a rotation of axes, $\mathbf{A}\cdot\mathbf{B}$ is invariant or unchanged, because the lengths and the *relative angle* are unchanged by a rotation of axes.

Of course, for every proof with one definition, a possibly more cumbersome one, which uses the other definition, also exists.

Let us return to the work-energy theorem using the dot product notation:

$$dK = \mathbf{F}\cdot d\mathbf{r} = dW. \tag{6.58}$$

The work done by a force when it moves a body by a vector $d\mathbf{r}$ is the length of the force vector times the distance traveled, times the cosine of the angle between the force vector and the displacement vector. That is also the change in kinetic energy dK. Let us make a big trip in the $x-y$ plane, shown in Figure 6.3, starting from a point $\mathbf{r}_1 \equiv 1$ and ending at $\mathbf{r}_2 \equiv 2$, and made up of a sequence of little segments $d\mathbf{r}$ in each one of which I calculate $\mathbf{F}\cdot d\mathbf{r}$. When I add their contributions to the change in K and the work done, I get, *as the segments' sizes tend to zero*,

$$\int_1^2 dK = K_2 - K_1 = \int_1^2 \mathbf{F}\cdot d\mathbf{r}. \tag{6.59}$$

The right-hand side is called the *line integral of the force* **F** *between* 1 *and* 2 *along a path P*.

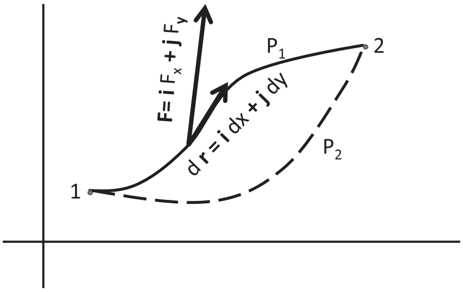

Figure 6.3 The line integral of a force between points 1 and 2 along a path P_1 is the sum of dot products $\mathbf{F} \cdot d\mathbf{r}$ over tiny segments that make up the path, in the limit $d\mathbf{r} \to \mathbf{0}$. Also shown by a dotted line is another path P_2 between the same end points.

6.4 Conservative and non-conservative forces

Suppose it is true, just like in one dimension, that the line integral of the force is something that depends only on the end points. Let us call the answer $U(1) - U(2)$, just like we did in one dimension. I am done, because then I have

$$K_2 + U_2 = K_1 + U_1. \tag{6.60}$$

To make sure this is correct, I ask the mathematicians a question: "You told me the integral of $F(x)$ from start to finish is really the difference of another function G at the upper limit minus G at the lower limit, with G related to F by $F = \frac{dG}{dx}$. Is there a similar result in $d = 2$?" Sadly, this is not the case. What could go wrong? Yes, friction will do it, but let us assume there is no friction, and that $\mathbf{F}$ depends only on $\mathbf{r}$. Can something still be wrong? Well, let me ask you the following question. Suppose I go from 1 to 2 along path P_1 and another person goes along path P_2. Do you think that person will do the same amount of work, even though the force is now integrated on a longer path? In two dimensions, there are thousands of ways to go from one point to another point. Therefore, this integral is not specified by just the end points; it depends on the entire path, which needs to be specified. If the work done depends on the path, then the answer cannot be of the form $U(1) - U(2)$, which depends only on the end points.

I digress to point out that even in $d = 1$, there are many ways to go from x_1 to $x_2 > x_1$. For example, we can go directly to x_2 or we can overshoot to x_3 and swing back to x_2. The answer will be the same, because for every segment from x_2 to x_3 that makes a contribution $F(x)dx$, an equal and opposite contribution exists on the way back to x_2, because $F(x)$ remains the same, and dx changes sign. In this sense, every force $F(x)$ in $d = 1$ is conservative. Of course, if it is friction we are talking about, the two canceling pieces now add, because $F = F(x, v(x))$ reverses sign along with dx.

Returning to $d = 2$, I am going to show that the work done by a generic force will be path-dependent. To generate a random force, I asked my class to give me numbers from 1 to 3, and I got the following list: 2, 2, 2, 1, 1, and 2. Using these randomly generated numbers as coefficients and exponents, I wrote down a force:

$$\mathbf{F}(x, y) = \mathbf{i}2x^2y^2 + \mathbf{j}xy^2. \tag{6.61}$$

For example, the $2x^2y^2$ is from the first three 2's chosen by the class.

Is it true for this generic force, essentially picked out of a hat, that the work done in going from one point to another depends only on the end points, or does it depend in detail on how you go between the end points? We will find that the work done along two paths, joining the same two end points, will give two different answers.

Let's find the work done in moving from the origin, $(0, 0)$, to the point $(1, 1)$. I will take the two paths shown in Figure 6.4. In one path I go horizontally until I'm at $(1, 0)$, below the point $(1, 1)$, and then straight up to $(1, 1)$. In the other path, I'm going straight up to $(0, 1)$ and then on horizontally to $(1, 1)$. So, let's find the work done when I go the first way. I'm

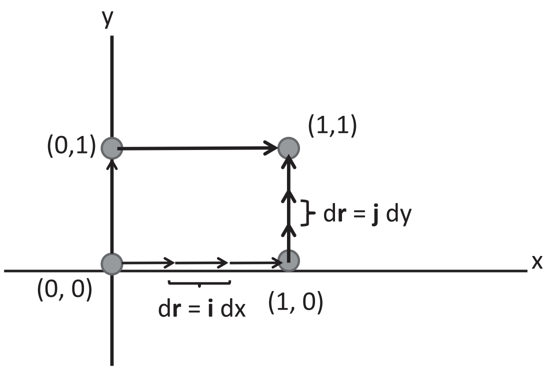

Figure 6.4 The line integral of a vector from $(0, 0)$ to $(1, 1)$ along two paths.

going to integrate $\mathbf{F}\cdot d\mathbf{r}$ first on the horizontal segment, then on the vertical segment. On the x-axis if I move a little bit I have

$$d\mathbf{r} = \mathbf{i}dx \tag{6.62}$$
$$\mathbf{F}(x,y) = \mathbf{i}2x^2y^2 + \mathbf{j}xy^2 = 0 \text{ because } y=0 \text{ on the } x\text{-axis} \tag{6.63}$$
$$\mathbf{F}\cdot d\mathbf{r} = 0. \tag{6.64}$$

In other words, the work done in this segment is zero because $\mathbf{F}$ itself vanishes when $y=0$. In the vertical segment from $(1,0)$ to $(1,1)$,

$$d\mathbf{r} = \mathbf{j}dy \tag{6.65}$$
$$\mathbf{F}(x,y) = \mathbf{i}2x^2y^2 + \mathbf{j}xy^2 = \mathbf{i}2y^2 + \mathbf{j}y^2$$
$$\text{because } x=1 \text{ on this segment} \tag{6.66}$$
$$\mathbf{F}\cdot d\mathbf{r} = y^2\,dy \tag{6.67}$$
$$\int \mathbf{F}\cdot d\mathbf{r} = \int_0^1 y^2\,dy = \frac{1}{3}. \tag{6.68}$$

So the work done on this path is $W_1 = 0 + \frac{1}{3} = \frac{1}{3}$.

On the second path, we have no contribution from the vertical segment because $\mathbf{F}=0$ for $x=0$. In the horizontal segment at $y=1$, we have

$$d\mathbf{r} = \mathbf{i}dx \tag{6.69}$$
$$\mathbf{F}(x,y) = \mathbf{i}2x^2y^2 + \mathbf{j}xy^2 = \mathbf{i}2x^2 + \mathbf{j}x$$
$$\text{because } y=1 \text{ on this segment} \tag{6.70}$$
$$\mathbf{F}\cdot d\mathbf{r} = 2x^2\,dx \tag{6.71}$$
$$\int \mathbf{F}\cdot d\mathbf{r} = \int_0^1 2x^2\,dx = \frac{2}{3}. \tag{6.72}$$

So the work done on this path is $W_2 = 0 + \frac{2}{3} = \frac{2}{3}$.

The answer is path-dependent.

I have shown you that if we took a random force, the work done is dependent on the path. For this *non-conservative force*, you cannot define a potential energy, whereas in one dimension any force other than friction allowed you to define a potential energy.

Our quest for a conserved energy leads us to search for a *conservative force*, a force for which the work done in going from 1 to 2 is path-independent.

6.5 Conservative forces

At first sight a conservative force looks miraculous. A randomly generated force was seen to have a line integral that depended on the path. How can the path dependence ever go away? Do conservative forces exist, and, if yes, how are we to find them?

Do not despair. Here is an algorithm that will produce any number of conservative forces.

- Take any function $U(x,y)$.
- The corresponding conservative force is

$$\mathbf{F} = -\mathbf{i}\frac{\partial U}{\partial x} - \mathbf{j}\frac{\partial U}{\partial y}. \tag{6.73}$$

- The potential energy associated with this conservative force will be U itself.

Here is an example.

$$U(x,y) = xy^3 \tag{6.74}$$

$$\frac{\partial U}{\partial x} = y^3 \tag{6.75}$$

$$\frac{\partial U}{\partial y} = 3xy^2 \tag{6.76}$$

$$\mathbf{F} = -\mathbf{i}y^3 - \mathbf{j}3xy^2. \tag{6.77}$$

Let me prove to you that the recipe works. The change in the function U, due to a small deviation from (x,y) to $(x+dx, y+dy)$, is

$$dU = \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \tag{6.78}$$

in the limit as all changes go to zero. Writing this in terms of **F**

$$dU = -F_x dx - F_y dy = -\mathbf{F} \cdot d\mathbf{r}. \tag{6.79}$$

Adding all the little pieces and changing the sign of both sides, we get

$$U(1) - U(2) = \int_1^2 \mathbf{F} \cdot d\mathbf{r} = K_2 - K_1, \tag{6.80}$$

which is the law of conservation of energy with U as the potential energy.

So I cooked up a force such that $\mathbf{F} \cdot d\mathbf{r}$ was a change in a certain function U. If I add all the $\mathbf{F} \cdot d\mathbf{r}$'s, I'm going to get the change in the function U from start to finish. We are beginning to see why certain integrals do not depend on the path. Here is an analogy. Forget about integrals. Imagine I am on some hilly terrain. I start at one point, and I walk to another point. At every portion of my walk, I keep track of my change in altitude, with uphill as positive and downhill as negative. That is like my dU. I add them all up. The total height change will be the difference in the heights of the end points. Now, you start with me but go on a different path. You wander all over the place but finally stop where I stopped. If you kept track of how long you walked, it won't be the same as my walk. But if you also kept track of how many feet you climbed at each step and added them all up, you would get the same answer I got. I repeat: if what you were keeping track of was the height change in a function, then the sum of all the height changes will simply be the height at the end minus the height at the beginning, independent of the path. Conversely, starting with the height function, if you manufacture a force **F** whose components are its partial derivatives, $\mathbf{F} \cdot d\mathbf{r}$ will measure the height change in each segment, and the line integral will yield the total height change between start and finish, independent of the path.

Consider the line integral of a conservative force on a *closed loop*, that is, when the starting and ending points 1 and 2 in Figure 6.3 coincide. Because this represents the change in U between some point and the *same* point, it vanishes for any loop. This is expressed as follows:

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad \text{if } \mathbf{F} \text{ is conservative.} \tag{6.81}$$

Are there other ways to manufacture the conservative force? No! One can show that every conservative force can be obtained by differentiating some corresponding U.

Remember how I went out on a limb with the randomly chosen force the class generated and promised I was going to do the integral along two paths and get two different answers? What if the force had been a conservative force? Then I would have been embarrassed, because I would find, after all the work, that both paths gave the same answer. So, I had to make sure right away that the force was not conservative. How could I tell? I asked myself, "Could there be some function U (the negative of) whose x and y derivatives could equal $\mathbf{i}2x^2y^2 + \mathbf{j}xy^2$?" I knew the answer was No because if I took a y derivative of such a U to get F_y, then F_y should have one less power of y than F_x, but in our example the powers of y were the same in both. I will describe shortly a better way to analyze this question.

While it is true that even one example of path dependence (as illustrated above) is enough to show a force is non-conservative, getting the same answer on two or even two thousand paths between any number of fixed end points does not mean the force is conservative. It could be accidental. Some other path or some other end points may show the force is non-conservative. Conversely it could happen that a non-conservative force, like the one I just worked with, has the same integral for two particular paths joining two particular end points by pure accident. I took that gamble and lucked out.

But if the force is really conservative, how are we to show that? Here is the wonderful *test* I promised. If $\mathbf{F}$ is conservative, it must come from a U by taking partial derivatives, as per Eqn. 6.73. It follows that

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 U}{\partial y \partial x} \tag{6.82}$$

$$\frac{\partial F_y}{\partial x} = -\frac{\partial^2 U}{\partial x \partial y} \quad \text{which means} \tag{6.83}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \quad \text{because the cross derivatives are equal.} \tag{6.84}$$

If I give you a force and ask you, "Is it conservative?" you simply see if

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}. \tag{6.85}$$

If it is, you know the force is conservative; if not, it is not.

The example we considered,

$$\mathbf{F}(x, y) = \mathbf{i}2x^2y^2 + \mathbf{j}xy^2, \tag{6.86}$$

fails the test:

$$\frac{\partial F_x}{\partial y} = 4x^2y \neq \frac{\partial F_y}{\partial x} = y^2. \tag{6.87}$$

The two most ubiquitous forces, gravitational and electrostatic, are conservative.

For longer discussion of this topic that fills in many blanks, see my *Basic Training in Mathematics.*

6.6 Application to gravitational potential energy

Let's take the most popular example: the force of gravity near the surface of the earth given by $\mathbf{F}_g = -\mathbf{j}mg \equiv m\mathbf{g}$ where $\mathbf{g} = -\mathbf{j}g$. It is conservative because the x derivative of F_y vanishes, and there is no F_x to differentiate, so that $\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} = 0$. What is the potential U that led to this? You can easily guess that $U = mgy$ will obey $F_y = -\frac{\partial U}{\partial y}$. You can also have $U = mgy + 96$, but we will not add those constants. In the law of conservation of energy, $K_1 + U_1 = K_2 + U_2$, adding a 96 to the U on both sides doesn't do anything. You already knew this from our study of motion in one dimension, and I am pointing out that this is also true in two dimensions.

Consider an application. Figure 6.5 shows a roller-coaster track that has a wiggly shape. At every x, there's a certain height $y(x)$ and a potential energy $U(x) = mgy(x)$, which is essentially just the profile of the roller-coaster track. If a coaster begins at rest at point A at the top, what is its total energy? It has a potential energy given by the height h, it has no kinetic energy, and so the total energy is just $E_1 = mgh$. But the total energy cannot change as the coaster goes up and down. So, you draw a line at height E_1 to represent this total energy. If the coaster is at some point x, then $U_1(x)$ is its potential energy and the rest of E_1 is its kinetic energy $K_1(x)$ as shown. As it oscillates up and down during its ride, the coaster gains and loses kinetic and potential energies, which always add up to the same E_1.

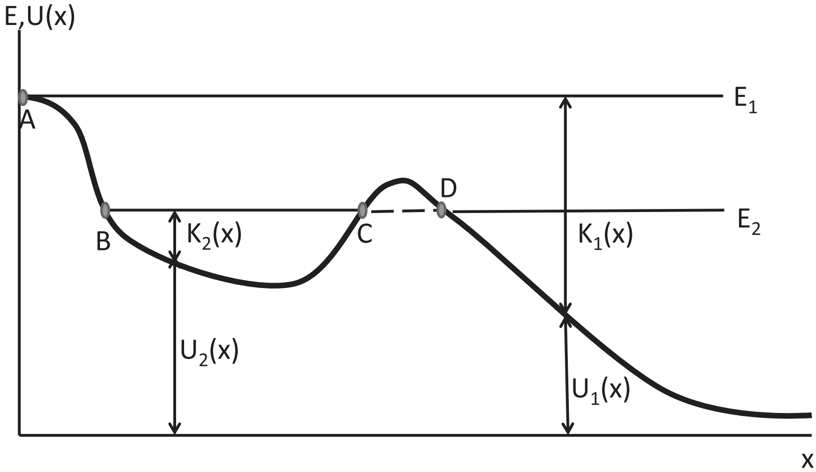

Figure 6.5 The roller-coaster ride. The total energy is fixed at E_1 or E_2 in the two examples discussed. At every point x, the sum of the potential energy $U(x)$ and the kinetic energy $K(x)$ equals a constant E. If the energy is E_2, the coaster can only be found between B and C or to the right of D. It is disallowed in the region CD where the potential energy exceeds the total energy, and K would have to be negative.

Consider a roller coaster whose total energy is E_2. We release it from rest at point B. It'll come down, pick up speed, slow down, stop, and turn around at C, because, at that point, the potential energy is equal to the total energy and there is no room for any kinetic energy. It'll rattle back and forth between B and C. If we release it from rest at D, it will have the same energy E_2, and it will coast down to the end of the ride. But it can never go from C to D because in the region CD it would have more potential energy than total energy, and hence negative kinetic energy, which is impossible.

However, according to laws of quantum mechanics, a particle with energy E_2 can disappear from the region BC and tunnel to D with the same energy. I use the word *tunnel* because in classical mechanics, the particle cannot cross the potential energy barrier in the interval CD. In quantum theory you cannot raise this objection because particles do not move along continuous, interpolating trajectories between two observed locations.

Back to the coaster: We can use energy conservation to find the speed at any point along the track. We can use it to determine the minimum height H from which the coaster in Figure 4.6 must be released so as to reach the top of the loop (at a height $2R$ measured from the ground) at the minimum requisite speed of $v = \sqrt{Rg}$. We write

$$\frac{1}{2}m \cdot 0^2 + mgH = \frac{1}{2}mRg + mg(2R) \tag{6.88}$$

$$H = \frac{5}{2}R. \tag{6.89}$$

A final note. The law of conservation of energy for the coaster as I stated it is incomplete, because gravity is not the only force acting. There is $\mathbf{F}_T$, the normal force of the track. Look, if I didn't want to have any force but gravity, I could take this roller coaster and just push it over the edge of a cliff. That converts potential to kinetic energy, but the outcome is not going to be good for the riders. Park designers build a track because they want the customers to survive the ride and come back for more. So the track should exist, and the consequent $\mathbf{F}_T$ should be included in computing the work. Luckily, this normal force does no work, because $\mathbf{F}_T \cdot d\mathbf{r} = 0$ in every portion. So the correct thing to do would be to say $K_2 - K_1$ is the integral of all the forces, divide them into $\mathbf{F}_T$ due to the track and $\mathbf{F}_g$ due to gravity, and drop $\mathbf{F}_T$ for the reason mentioned.

Selected Problems

Exercise 4.4 Are the following fields conservative? If yes, what are the potentials $U(x,y)$? (i) $\mathbf{F} = \mathbf{i}y\sin x + \mathbf{j}\cos y$, (ii) $\mathbf{F} = \mathbf{i}2xy^3 + \mathbf{j}3y^2x^2$, (iii) $\mathbf{F} = \mathbf{i}\sin x + \mathbf{j}\cos y$, and (iv) $\mathbf{F} = \mathbf{i}\cosh x \cosh y + \mathbf{j}\sinh x \sinh y$.

Exercise 4.9 A mass m falls from a height h onto a spring of force constant k. Show that the maximum spring compression is $(mg/k)\left(1+\sqrt{1+2kh/mg}\right)$. Show that the maximum speed attained is $v_{max} = \sqrt{2gh}\left(1+\frac{mg}{2kh}\right)$. Hint: At what x does it stop accelerating downward?

Exercise 4.14 A particle of mass m is in a potential $V(x) = ax^2 + bx^4$. (i) Find its minima if $a, b > 0$. (ii) Repeat if $a = -|a|$. (Plot the function to see how many minima there are.) (iii) Find the frequency ω of small oscillations about the minima.

Exercise 4.15 A particle of mass m is in a periodic potential $V(x) = -A\cos(2\pi x)$. (i) Where are its minima? (ii) What is the frequency of small oscillations about a minimum?

Exercise 4.16 Find the work done by a force $F = \mathbf{i}x^2y^3 + \mathbf{j}x^3y^2$ acting on a body from (0, 0) to (1, 1) along three paths: (i) first along x and then along y, (ii) at 45°, and (iii) along the curve $y = x^2$. (iv) Show that this is a conservative force and find the potential energy $U(x;y)$. Find the work done earlier in terms of difference in U. (v) Repeat parts (i) and (ii) for $\mathbf{F} = \mathbf{i}xy^3 + \mathbf{j}xy$.