An Elementary Derivation of the Navier-Stokes Equation and the Sound Wave Equation

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Abstract

The Navier-Stokes equation, which is the central equation of fluid dynamics, is seldom though at the undergraduate level, let alone in an introductory physics class. As a consequence, actual fluid dynamics at the professional level looks nothing like the introductory physics fluid dynamics, and the students do not get a taste of what the discipline is all about and whether they would like to pursue it. This manuscript is an attempt to derive the Navier-Stokes equation, and the wave equation for sound waves, at the level of a Calculus base introductory physics class.

The way to mathematically analyze a fluid is by considering a differential volume-element, or a *parcel* (see Fig. 1), and considering the forces acting on this parcel. First, I will recount how this method gives us the pressure equation for a liquid at equilibrium in a gravitational field (Fluid Statics); later, I will derive the Navier-Stokes equation with similar considerations.

1 Fluid Statics

Consider Figure 1. This represents a differential fluid element, or parcel, in the form of a cube with sides dx, dy, and dz. Such that the volume if given by $dV = dxdydz$. In a static situation, this cube does not move. This means that the pressure—which is a product of the particles in the fluid colliding with the walls on the cube due to their random (thermal) motion—cancels out the effects of gravity. Gravity's nature is to bring the fluid parcel downwards (towards the center of the Earth), so the net force due to pressure must be upward. We hence have, from Newton's second law:

$$Fz_{\text{total}} = -m_{\text{parcel}}g + \Delta_z P * dxdy = 0 \tag{1}$$

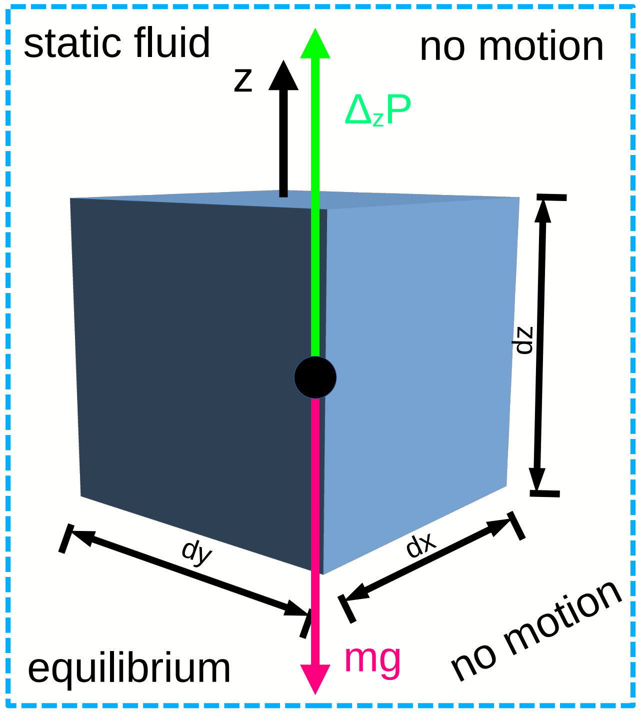

Figure 1: A (zoomed-in) differential volume element (or parcel) on a static fluid. For the fluid to be in static equilibrium, an upward force, produced by the pressure of the surrounding fluid on the parce, must balance the weigth of the parcel

where g is the gravitational acceleration, P is the pressure, and Δ_y represents the change in pressure between the top and bottom faces of the cube. In a fluid, pressure exerts a net force on a fluid parcel when there is a *pressure difference* between opposite faces. The change in pressure along a length dz (from one face to the other) is given by the derivative of the pressure in the z-direction, times the length dz. To see this, consider that the pressure in the bottom face is P_0, then by a first-order Taylor expansion, the pressure in the top face will be $P_0 + \frac{dP}{dz}dz$. Here, we are assuming the pressure to vary linearly within the cube, which is a good approximation given that the cube is infinitesimal. $\Delta_y P$ is then given by the difference in these two pressures:

$$\Delta_z P = P_{\text{bottom}} - P_{\text{top}} = P_0 - (P_0 + \frac{dP}{dz}dz) = -\frac{dP}{dz}dz$$

Eqn. 1 then becomes:

$$Fz_{\text{total}} = -m_{\text{parcel}}g - \frac{dP}{dz}dzdxdy = 0 \tag{2}$$

Dividing by the differential volume dV and recognizing that $\frac{m_{\text{parcel}}}{dV} = \rho$ (where ρ is the density of the fluid) we find that

$$\frac{dP}{dz} = -\rho g \tag{3}$$

If we consider an ambient pressure P_{atm} over a liquid, we might want to know how the pressure will change when descending a depth $-d$ into the fluid (the negative is because the displacement is downwards). Integrating this equation we find that

$$\int_{P_{\text{atm}}}^{P} P' = -\int_{0}^{-d} \rho g dz$$

$$P = P_{\text{atm}} + \rho g d \tag{4}$$

This is how the pressure increases as we dive further into the sea or a pool of water.

2 Fluid Dynamics: The Navier-Stokes Equation

This is similar to what we did when doing fluid statics: we derived the pressure-depth relationship by considering a fluid parcel in equilibrium in a fluid container. Now the parcel can have a net velocity and acceleration, so the analysis gets more involved, but we still use the same methodology as in fluid statics.

Consider the volume element in Fig. 2. What are the forces in the volume-element? These will be due to the neighboring particles (neglecting gravity for now). There are two types of forces neighboring particles will exert on the fluid parcel. One is collisional. The particles will collide with the faces of the cube and exert a pressure on the cube, pushing in on the walls of the cube. Pressure captures the forces due to the collisions given the random (thermal) motions of the fluid. Like the case of electrostatic, the net influence of pressure depends on how its difference between the faces of the cube. While in the electrostatic case there was only a pressure difference in the vertical z-axis (because this difference was balancing gravity's influence), now there can be a pressure difference in the x, and y-axis as well. Hence:

$$F_{Px} = \Delta_x P = (-\frac{\partial P}{\partial x}dx)dydz \tag{5}$$

$$F_{Py} = \Delta_y P = (-\frac{\partial P}{\partial y}dy)dxdz \tag{6}$$

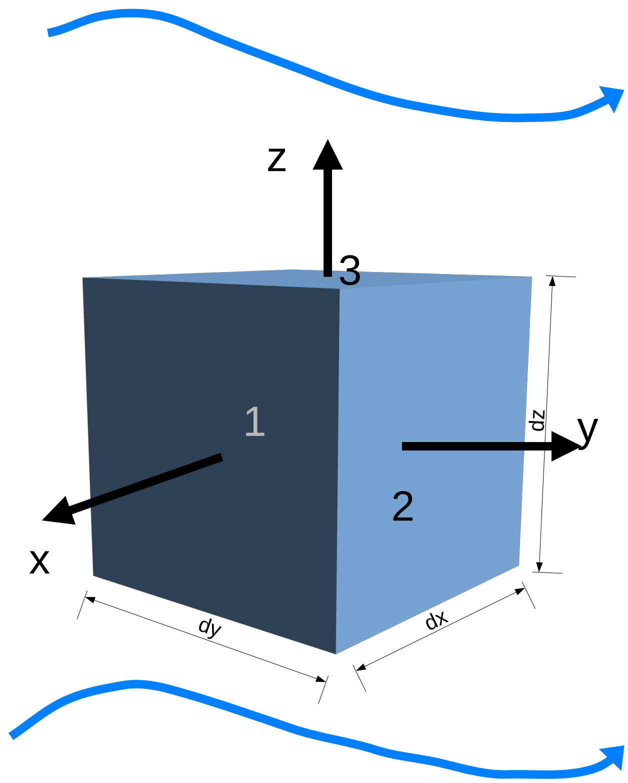

Figure 2: A differential volume-element in a flow. Each face of the cube is labeled with a number. The face normal to the x-axis is 1, to the y-axis is 2, and to the z-axis is 3. The opposite faces will be refered to as -1, -2, and -3, respectively.

$$F_{Pz} = \Delta_z P = (-\frac{\partial P}{\partial z} dz) dx dy \tag{7}$$

where we have multiplied by the area of each face ($dydz$, $dxdz$, and $dxdy$, respectively) to obtain the net force on those faces (recall that force in pressure times the area). The negative sign comes about because, when the pressure increases in one direction, the net force will be opposite to this direction (as the higher-pressure side will be pushed opposite to this pressure increase).

The second force is attractive and related to drag or *viscosity*. Inter-molecular forces want to stick fluid particles together, so if a fluid molecule passes by another one, this will feel an attractive force towards the passing particle. This effect is called viscous shear.

Consider, for instance, face 2 of the cube. There are four possible interactions with the surrounding particles that will produce a force on this wall, which are shown in Fig. 3. The cyan, magenta and green particles are passing by the wall and exerting an attractive inter-molecular force that is represented as a viscous drag. Let's consider the force due to the magenta particle. A net force on the cube due to magenta particles (particle flowing in the +y direction) will only exist if the magenta particles are changing their velocity with respect to the parcel. If they are at the same velocity, then no drag will be experienced (the way this inter-molecular attraction works, is that the parcel's fluid particle only wants to follow the particles that pass by them). To compure the force on face 2 (F_{M2}), we first compute what the force would be in the center of the parcel. The force due to magenta particles at the center (F_0) depends on the difference in the y velocity of the fluid (over a small distance δy close to the center) with respect to the velocity of the parcel (referred to as $v_y(0)$)

$$F_{M0} = \nu(\frac{v_y - v_y(0)}{\delta y})dxdz \rightarrow \nu\frac{\partial v}{\partial y}dxdz \tag{8}$$

where ν is a constant of proportionality called the viscosity, which is related to the strength of the attractive inter-molecular forces. The force on face 2 would then be given by extrapolating (via a Taylor expansion) a distace $\frac{dy}{2}$:

$$F_{M2+} = \nu(\left.\frac{\partial v_y}{\partial y}\right|_0 + \frac{\partial^2 v_y}{\partial y^2}\frac{dy}{2})dxdz \tag{9}$$

Even if this difference in velocity exists, that is not enough to warrant that the magenta particles will exert a net force on the *cubic parcel*. If the same force F_{M2+} is being applied in the opposite face, the parcel will not experience a drag. This is because for a fluid parcel, what matters is a difference between the velocity of the magenta particle in face 2 *versus it's opposite face*. If both faces have the same pressure (force per area), the cube will not accelerate. In general, the acceleration of the fluid parcel is dictated by the difference in pressure between two opposite faces of the parcel. Note that to find the force on the opposite face (F_{M2-}) we extrapolate the force in F_{M0} a distance $-\frac{dy}{2}$ to the -2 face.

$$F_{M2-} = \nu[\left.\frac{\partial v_y}{\partial y}\right|_0 + \frac{\partial^2 v_y}{\partial y^2}(\frac{-dy}{2})]dxdz$$

Hence, the force due to the magenta particles interacting with face 2 and its opposite will be given by this difference:

$$F_{M2} = F_{M2+} - F_{M2-} = \nu[\overbrace{\overbrace{(\left.\frac{\partial v_y}{\partial y}\right|_0 + \frac{\partial^2 v_y}{\partial y^2}\frac{dy}{2})}^{\text{parcel's and particle's velocity difference}} - (\left.\frac{\partial v_y}{\partial y}\right|_0 + \frac{\partial^2 v_y}{\partial y^2}\frac{(-dy)}{2})}^{\text{change of this difference between the two parallel faces of the cube}}]\,dxdz \tag{10}$$

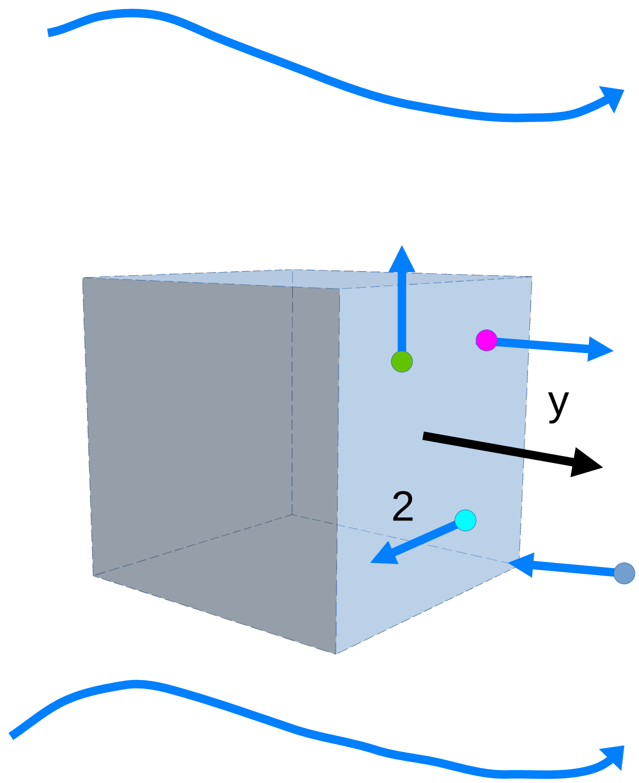

Figure 3: The four types of interaction that neighboring fluid particles can have on face 2 (or any other face) of the fluid parcel. The Cyan, magenta, and green particles exert viscous intermolecular forces that, given that this interaction is not the exact same on the opposite face, will cause a net acceleration in their direction of motion. Meanwhile, the blue force collides with face 2 exerting a pressure

where, again, ν is the viscosity coefficient of the fluid (which depends on the streght of this inter-molecular flow) and where the terms $(v_y(0) \pm \nu \frac{\partial v_y}{\partial y}|_0 \frac{dy}{2})$ represent the change in velocity between the cube and the magenta particles on both faces of cube. It is the difference between this change (hence, the second derivative) that matters. ν relates the pressure on the faces of the parcel to the shear, so, to obtain the force, we have to multiply by the area of the face $dxdz$ (recall that force is pressure times the area). Simplifying this equation, we get:

$$F_{M2} = \nu \frac{\partial^2 v_y}{\partial y^2} dydxdz \tag{11}$$

Now, consider the forces exerted by the green (cyan) particle on face 2. The particle flows upwards, so the viscous (attractive intermolecular force) will be upwards. But this force will only exist if there is a difference in velocity between face 2 and the green particle, and if this difference itself changes between face 2 and its opposite face. So we are left with the force:

$$F_{G2} = \nu[\overbrace{\overbrace{(\frac{\partial v_z}{\partial y}\Big|_0 + \frac{\partial^2 v_z}{\partial y^2}\frac{dy}{2})}^{\text{parcel's and particle's velocity difference}} - (\frac{\partial v_z}{\partial y}\Big|_0 + \frac{\partial^2 v_z}{\partial y^2}\frac{(-dz)}{2})}^{\text{change of this difference between the two parallel faces of the cube}}]\, dxdz \tag{12}$$

This again simplifies to:

$$F_{G2} = \nu \frac{\partial^2 v_z}{\partial y^2} dydxdz \tag{13}$$

Unsuprisingly, by the same argument, the net force due to the cyan particles acting on face 2 and its opposite ends up being:

$$F_{C2} = \nu \frac{\partial^2 v_x}{\partial y^2} dydxdz \tag{14}$$

Note that the direction of the force due to the shear is the same as the direction of motion of the particles. Magenta particles create a force in the y-direction, cyan in the x, and green in the z. While we considered only the forces in face 2, and its opposite, the net force on the cube will be a consequence of adding all the forces on all the faces of the cube. Magenta particles, as they "slide" on faces 1 and 3, will also produce a force in the y. Conversely, for the green and magenta particles. The net force on the y direction will be given by:

$$F_{\text{net}_y} = -\overbrace{\frac{\partial P}{\partial y}dV}^{\text{difference in pressure}} + \nu(\overbrace{\frac{\partial^2 v_y}{\partial y^2}dV}^{\text{cyan particles on face 2}} + \overbrace{\frac{\partial^2 v_y}{\partial x^2}dV}^{\text{cyan particles on face 1}} + \overbrace{\frac{\partial^2 v_y}{\partial z^2}dV}^{\text{cyan particles on face 3}}) = \frac{\partial}{\partial t}(m_{\text{parcel}} v_y) \tag{15}$$

where we have equated the net force in the y to the net change in momemtum in the y (this is Newton's 2nd Law). Dividing the right-hand side by dV (the fluid parcel volume):

$$\frac{\partial}{\partial t}(\frac{m_{\text{parcel}}}{dV}v_y) = -\frac{\partial P}{\partial y} + \nu(\frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2}) \tag{16}$$

and recognizing that $\frac{m_{\text{parcel}}}{dV}$ in the density ρ of the fluid, we get the y component of the Navier-Stoke equation.

$$\rho\frac{\partial v_y}{\partial t} = -\frac{\partial P}{\partial y} + \nu(\frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial z^2}) \tag{17}$$

where by taking the the density of the fluid ρ out of the partial derivative in the left-hand side of the equation we are taking the density to never change with time. This is true for *incrompressible* fluids, like water. We will relax this assumption when dealing with air and sound waves, as air is compressible. This equation, however, is a very powerful and general tool to describe the behaviors of water, which is incompressible, and no matter the velocity, the density is always $\rho_{H^2O} = 1000kg/m^3$.

3 The fictitious force

A final caveat, however, before we write all the components of the Navier-Stokes equation. Care must be taken when considering the acceleration of the fluid parcel $\frac{dv_y}{dt}$. This acceleration is an acceleration with respect to the frame shown in Fig. 4, which is a frame that "rides" with the fluid parcel itself, keeping itself at the center of the parcel always. This means that this frame is not necessarily an inertial frame. In general, when the frame is not inertial, Newton's second law can be written as:

$$\boldsymbol{F} - m\boldsymbol{a}_{\text{frame}} = m\boldsymbol{a} \tag{18}$$

where $-m\boldsymbol{a}_{\text{frame}}$ is a fictitious force that arises due to the non-inertial effects of the frame. The final step in our derivation will hence be to add to Equation 17 a fictitious force term $-ma_{\text{frame}_y}$, the y-component of this fictitious force. We compute a_{frame_y} by considering that all the frame is doing, to keep itself on top of the fluid particle, is following the flow of the fluid $\boldsymbol{v}$. Consider that at a time intercal dt, the frame moves a distance ds along the direction of the flow. What is the change of the v_y when moving along this flow a distance ds? This is given by:

$$a_{\text{frame}_y} dt = \frac{\partial v_y}{ds} ds \tag{19}$$

To obtain the rate of this change on a time interval dt, we devide both sides by dt and find that:

$$a_{\text{frame}_y} = \frac{\partial v_y}{ds}\frac{ds}{dt}$$

where $\frac{ds}{dt}$ is the speed of the flow $|v|$ and the derivative $\frac{d}{ds}$ is a derivative in the direction of the velocity flow. Normally derivatives are taken in the x, y or , direction, but one can take a derivative in any direction via the *directional derivative*. This is defined as:

$$\frac{d}{ds} = (\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}) \cdot \frac{\boldsymbol{v}}{|\boldsymbol{v}|}$$

where $\frac{\boldsymbol{v}}{|\boldsymbol{v}|}$ yields the unit vector in the direction of the flow. Substituting this into the expression

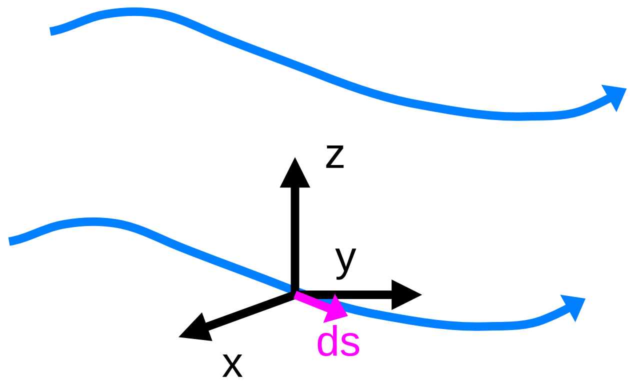

Figure 4: In a time interval dt the frame that follows the fluid parcel travels a differential length ds in the direction of the flow. This will create non-inertial effect on the motions described with respect to this frame. These effects are accounted for by addid a fictitious force.

$\frac{\partial v_y}{ds}\frac{ds}{dt}$ we get

$$\frac{\partial v_y}{ds}\frac{ds}{dt} = \left(\frac{\partial v_y}{\partial x}vx + \frac{\partial v_y}{\partial y}v_y + \frac{\partial v_y}{\partial z}v_z\right)\frac{\cancelto{1}{\frac{ds}{dt}}}{\cancel{\frac{ds}{dt}}}$$

where we have used the fact that $|v| = \frac{ds}{dt}$ So finally, we correct for the fact of a non-inertial frame by adding the term

$$-\rho a_{\text{frame}_y} = -\rho\left(\frac{\partial v_y}{\partial x}v_x + \frac{\partial v_y}{\partial y}v_y + \frac{\partial v_y}{\partial z}v_z\right)$$

to the right-hand side of Eqn. 17.

$$\rho\frac{dv_y}{dt} = -\frac{\partial P}{\partial y} + \nu(\frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}) - \rho[\frac{\partial v_y}{\partial x}v_x + \frac{\partial v_y}{\partial y}v_y + \frac{\partial v_y}{\partial z})v_z] \quad (20)$$

Again, this final term, in the square brakets, is a ficticios force (like the centrifugal force) that arrise because we considered a non-inertial frame which follows the fluid parcel around the flow. The x, and z components of the Navier-Stokes equations are similarly written as:

$$\rho\frac{dv_z}{dt} = -\frac{\partial P}{\partial z} + \nu(\frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2}) - \rho[\frac{\partial v_z}{\partial x}v_x + \frac{\partial v_z}{\partial y}v_y + \frac{\partial v_z}{\partial z}v_z] \quad (21)$$

$$\rho\frac{dv_x}{dt} = -\frac{\partial P}{\partial x} + \nu(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial z^2}) - \rho[\frac{\partial v_x}{\partial x}v_x + \frac{\partial v_x}{\partial y}v_y + \frac{\partial v_x}{\partial z}v_z] \quad (22)$$

Now we will simplify this equation by considering a steady-state two-dimensional flow, and finally simplify the equation further by considering a fluid with no viscosity ($\nu = 0$).

4 A steady-state one dimensional flow

In physics, "steady-state" refer to a set of equations that do not change in time. A steady-state of a fluid is similar to the notion of *static equilibrium* in mechanics: the acceleration in all directions is zero. Moreover, when a fluid's velocity only has component (v_y) and the flow velocity only changes in a direction perpendicular to this flow (the z direction) we have a *one dimenssional flow*. Note that this not mean the fluid itself does to extend in three dimensions, rather, it means the fluid only flows in two dimensions. Considering we have such a fluid, the Navier-Stokes equations reduce to:

$$0 = -\frac{\partial P}{\partial z} + \nu\frac{\partial^2 v_z}{\partial z^2} - \rho[\frac{\partial v_z}{\partial z}v_z] \quad (23)$$

or

$$\rho[\frac{\partial v_z}{\partial z}v_z] = -\frac{\partial P}{\partial z} + \nu\frac{\partial^2 v_z}{\partial z^2} \quad (24)$$

Note that, while the acceleration with respect to the non-inertial frame may vanish, the frame itself can be accelerated, so the fictitious force term survives. In this class, we will solve simple problems with these equations and with the following equation, which considers a fluid with no viscosity.

5 Invicid (conservative) fluids

If the fluid has no appreciable intermolecular forces (like a noble gas), we may consider it to be invicid; that is, it's viscosity $\nu = 0$. In this case the viscosity terms vanish and we are left with the equations.

$$\rho[\frac{\partial v_z}{\partial y} v_y + \frac{\partial v_z}{\partial z} v_z] = -\frac{\partial P}{\partial z} \tag{25}$$

$$\rho[\frac{\partial v_y}{\partial y} v_y + \frac{\partial v_y}{\partial z} v_z] = -\frac{\partial P}{\partial y} \tag{26}$$

These are called the Euclid equations for a two-dimensional flow. Note, however, that for gasses, in many situations, the incompressibility assumption fails, and we must consider that ρ can change. If ρ changes then the steady-state. An example of that lies in the next section. For a one-dimensional invicid steady-state flow, the Navier-Stokes reduces to:

$$\rho \frac{\partial v_z}{\partial z} v_z = -\frac{\partial P}{\partial z} - \rho g \tag{27}$$

where we have added (finally) the gravitational term ρg that acts in the negative z direction. Using the "reverse" chain rule we can simplify the left-hand term as

$$\rho \frac{\partial v_z}{\partial z} v_z = \frac{1}{2} \rho \frac{\partial v_z^2}{\partial z}$$

Now the equation can be easily integrated with respect to z:

$$\int \rho \frac{\partial v_z^2}{\partial z} dz = -\int \frac{\partial P}{\partial z} dz - \int \rho g dz$$

$$\frac{1}{2} \rho v_z^2 = -P - \rho g z + C$$

where C is just the constant of integration. Rearranged, this is Bernoulli equation, which stablishes that the following quantity in a constant along the direction of a one-dimensional, steady-state, invicid, incompressible fluid.

$$\frac{1}{2} \rho v_z^2 + P + \rho g z = C \tag{28}$$

This is a statement of the conservation of energy in a fluid.

6 Sound Waves

Thus far, we have only dealt with incompressible fluids. While this works extremely well for liquids, (particularly water). Gases, however, do then to be invicid (no viscosity). What follows is a derivation of the wave equation for a compressible invicid fluid that starts at equilibrium (at rest).

Since the fluid is by requirement initially at rest, out non-inertial term (the fictitious force) vanishes in this case. Parting from Eqn. 23, but changing this to be in the y-axis and setting the viscosity to zero, we get

$$0 = -\frac{\partial P}{\partial y} + \cancelto{0}{\frac{\partial^2 v_y}{\partial y^2}} - \rho[\cancelto{0}{\frac{\partial v_y}{\partial y} v_y}] \tag{29}$$

Note, however, that even if there is no average fluid flow, sound waves are not in a steady state: the wave itself means the fluid is sometimes accelerating. Hence, we must replace the 0 in the left-hand side by the acceleration term $\frac{\partial \rho v_y}{\partial t}$ (here we need to leave ρ inside of the time derivative because we are dealing with a compressible fluid).

$$\frac{\partial(\rho v_y)}{\partial t} = -\frac{\partial P}{\partial y} \tag{30}$$

For the next step we must consider two things: one is the conservation of mass constrains the change of ρ (we will see how). The other one is that thermodynamics (the ideal gas law) yields a relation between pressure P and density ρ which we will use.

6.1 The continuity equation (conservation of mass)

The density of a compressible fluid ρ can change in time and in space. For a one dimensional fluid, that means that it depends on the position of the fluid parcel along the y-axis, and the time t elapsed. But density cannot decrease out of thin air, if it decreases a point y in a time interval dt then this mass flowed with a velocity v_y though an area A into this a new point $y + dy$. The mass of the volume that flowed is:

$$M_{\text{flowed out}} = \rho A v_y dt \tag{31}$$

By conservation of mass, this equals the change of mass $d\rho$ of the parcel over a time dt. Consider the volume of this parcel to be $dV = Ady$

$$M_{\text{loss}} = d\rho A dy = -\frac{\partial \rho}{\partial t} dt A dy$$

By conservation of mass, these two expressions must be equal, we get:

$$\frac{\partial \rho}{\partial t} dtAdy = -\rho A v_y dt$$

Dividing both sides by dt and dy and noticing that the areas A cancel out, we see that conservation of mass implies that:

$$-\frac{\partial \rho}{\partial t} = \frac{\partial(\rho v_y)}{\partial y} \tag{32}$$

6.2 Ideal Gases

(TL;DR: In gases, pressure and density are related linearly by $P = \frac{c^2}{\gamma}\rho$ where γ is the adiabatic constant. The infinitesimal changes of preassure and density (dP and $d\rho$) are related by $dP = c^2 d\rho$)

Ideal gases (which we are superficially covered by the end of the class, and it is covered extensively in Chem 1) provide us with a relation between pressure and density, which we need to derive the wave equation from the Navier-Stokes Equation. Normally, the idea gas law is written as

$$PV = NRT$$

where R is Avogrado's number, N in the total number of moles in the gas, and T in the temperature. If we write it instead in terms of the total numbner of gas *molecules* N_m, we just need to use the Boltzmann constant k_B instead of Avogrados' number.

$$PV = N_m k_B T$$

We can divide both sides by V in order to obtain a relationship between *number density* (number of molecules per unit volume) and pressure

$$P = \frac{N_m}{V} k_B T$$

If we multiply both sides by the mass of the air molecule (which is ussualy the mass of the N_2 molecule as air on Earth is mostly hydrogen) we find that:

$$PM_{N_2} = \frac{M_{N_2} N_m}{V} k_B T$$

Note that the mass of each molecule multiplied by the total number of molecules N_m *is just the total mass of the gas*. Hence, the fraction if just ρ and we find that:

$$P = \frac{k_B T}{M_{N_2}} \rho \tag{33}$$

For what is to come, we do not only need a relation between P and ρ, but we also need a relation between the infinitesimal changes dP and $d\rho$ caused by a sound wave. To do this, we apply the differential on both sides of Eqn. 33.

$$dP = d(\frac{k_B T}{M_{N_2}}\rho)$$

Taking a differential is like taking a derivative so the product rule apply to the right hand side of the equation, were both density and temperature change with preassure.

$$dP = \frac{Tk_B}{M_{N_2}}d\rho + \frac{\rho k_B}{M_{N_2}}dT \tag{34}$$

To go futher we need to relate dT to dP. This relation generally depends on the type of process that is causing these changes. For the propagation of a wave, the pocess is adiabatic (e.i, it happens quickly compared to the hability of the air to transfer the heat). For thermodynamic reasons we won't go into, adiabaticity implies that the preassure and temperaature are related by:

$$P^{1-\gamma}T^{\gamma} = \text{Constant}$$

Taking the differential of this expression ($d(P^{1-\gamma}T^{\gamma})$) we get that:

$$P^{1-\gamma-1}(1-\gamma)T^{\gamma}d\rho + P^{1-\gamma}T^{\gamma-1}\gamma dT = 0$$

Dividing this by $P^{1-\gamma}T^{\gamma}$ in order to simplify, and isolating dT we get:

$$\frac{\gamma-1}{\gamma}\frac{dP}{P}T = dT$$

Plugging this into Eqn. 34 and simplyfying by using Eqn. 33, we finally get the relation:

$$dP = \gamma\frac{k_b T}{M_{N_2}}d\rho \tag{35}$$

We will simplify this equation by writting $\gamma\frac{k_b T}{M_{N_2}} = c^2$.

7 Derivation of the wave equation

We are now in position for deriving the wave equation combining Eqns. 30, 32, and 35. First, we need to derive Eqn. 32 with respect to time (remember that ρ is a function of time so it needs to be considered in the derivative as well).

$$-\frac{\partial^2 \rho}{\partial t^2} = \overbrace{\frac{\partial}{\partial t}\frac{\partial \rho v_y}{\partial y}} \tag{36}$$

Now we derive Eqn. 30 with respect to y

$$\overbrace{\frac{\partial}{\partial y}\frac{\partial \rho v_y}{\partial t}} = -\frac{\partial^2 P}{\partial y^2} \tag{37}$$

Note that the terms in the overbrace for both equations *are the same*: the order of derivation (first t and then y or viceversa) does not affect the result. So we can replace the left-hand side of Eqn. 37 simply with the term $\frac{\partial^2 \rho}{\partial t^2}$:

$$-\frac{\partial^2 \rho}{\partial t^2} = -\frac{\partial^2 P}{\partial y^2} \tag{38}$$

Finally, we replace $\partial^2 \rho$ with the change in preassure $\partial^2 P$ vía Eqn. 35 above in order to get an equation of only one variable (even if these are second order partial differentials, the derived relation between differentials still applies).

$$-\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = -\frac{\partial^2 P}{\partial y^2}$$

or

$$\frac{\partial^2 P}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = 0$$

Two derivatives in space minus two derivatives in time equals zero. *This is the wave equation*. It is the same equation we derived for waves on a string. Note that the general form of the wave equation is:

$$\frac{\partial^2 \psi}{\partial dt^2} + \frac{1}{v^2}\frac{\partial^2 \psi}{\partial y^2} = 0$$

By comparing these two last equations, we see that $c = v$. That is, the speed of the traveling pressure wave (sound wave) through a gas is given by $c = \sqrt{\gamma \frac{k_B T}{M_{N_2}}}$

8 A worked out example

> Water flows through a circular tube with inside diameter 0.2 m. A smoothly contoured cilinder with a hemispherical end of 0.15 m diamteter is held in the end of the tube where the water discharges to atmosphere. Neglect frictional effects and assume uniform velocity profiles at each section. Determine the relative pressure that the gauge will mesure (see the figure), and the force required to hold the body.

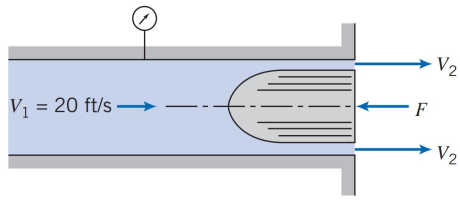

Figure 5: A cylindrical cap with a smooth hemispherical end is held at the end of an invicid pipe flow.

Given that we can assume uniform motion, we regard this as a one-dimmensional flow, where the velocity is in the x-axis. Moreover, there are no viscous forces, water is incompressible, and the fluid is at a steady-state. Hence we can use Bernulli equation in the x-axis.

$$\frac{1}{2}\rho v_x^2 + P = C \tag{39}$$

Conservation of mass (Eqn. 31) implies that the volume of water that flows from the left-hand-side of the figure in a time dt must be the same as the volume of water that flows at the rigth-hand-side out of the pipe:

$$\rho A_1 v_1 dt = \rho A_2 v_2 dt$$

Then v_2 is given by:

$$v_2 = \frac{A_1}{A_2} * v_1$$

We can compute C from Bernulli equation by considering that at the exit of the pipe, water "exits to atmosphere" so the preassure there is the atmospheric preassure P_{atm}.

$$\frac{1}{2}\rho(\frac{A_1}{A_2})^2 v_1^2 + P_{\text{atm}} = C$$

C is a constant of the motion so this will also be equal to:

$$\frac{1}{2}\rho v_1^2 + P_1 = \frac{1}{2}\rho(\frac{A_1}{A_2})^2 v_1^2 + P_{\text{atm}} = C$$

We are looking for the relative preassure $P_1 - P_{\text{atm}}$, which is equal to

$$P_1 - P_{\text{atm}} = \frac{1}{2}\rho v_1^2(\frac{A_1^2}{A_2^2} - 1) = \frac{\rho}{2} v_1^2(\frac{R^4}{(R^2 - a^2)^2} - 1) = \frac{1000}{2} 6.1^2 * (\frac{.1^4}{(.1^2 - .075^2)^2} - 1) = 78596.6\text{Pa} \tag{40}$$

where a is the radius of the hemisphere. The force required to hold the body will the same as the differneces in pressure, times the area where the pressures are applied (πa^2). The preassure inside the pipe, however, depends in the distance x from the left-end of the circle. If we consider a coordinate system at the center of the hemisphere, the velocity will be at a minimum for $x = -a$, and will become v_2 when $x = 0$. Consider $F = ma$ for this object; since it is in equilibrium $F_{\text{net}} = 0$. It would look something like this.

$$\int P(x) * dA - P_{\text{atm}} * A - F = 0 \tag{41}$$

where A is the crossectional area of the object (so, a circle with radius a) and where the preassue of the water balances the atmospheric preassure at the other end of the object plus the force. We have to integrate P because P depends on r (in changes at the pipe get tighter).

For now, consider the function (Eqn 40)

$$P = P_{\text{atm}} - \frac{1}{2}\rho v_1^2(\frac{A_1^2}{A_2^2} - 1) = P_{\text{atm}} - \frac{1}{2}\rho v_1^2(\frac{R^4}{(R^2 - r^2)^2} - 1)$$

where r^2 is the radius of the crossection of the cylinder. To find the force F, we must integrate the effects of this preassure from $r = 0$ to $r = a$ times the differential crossetional area of the hemisphere, which will just the be area of a circle:

$$P = \int_0^a (P_{\text{atm}} - \frac{\rho}{2}v_1^2(\frac{R^4}{(R^2 - r^2)^2} - 1))2\pi r dr$$

where we have replaced the differential area dA with the differential area of the circle $2\pi r dr$. This integral can be solved using an u-substitution of $u = R^2 - r^2$.

$$P = P_{\text{atm}} - \frac{\rho}{2}v_1^2\pi a^2 + \int_{R^2}^{R^2-a^2} \pi R^4 \frac{\rho}{2}v_1^2(\frac{1}{u^2})du$$

Here we only need the integral of u^{-2}. Solving and plugging in the integration limits we get:

$$P = P_{\text{atm}} + \pi\rho R^4\frac{v_1^2}{2}(\frac{1}{R^2} - \frac{1}{R^2 - a^2} - \frac{a^2}{R^4})$$

Plugging this back into Eqn . 41 and solving for F we find that:

$$F = \pi 1000\frac{6.1^2}{2}.1^4(\frac{1}{.1^2} - \frac{1}{.2^2 - .0.75^2} - \frac{.075^2}{.10^4}) = 255.582N$$