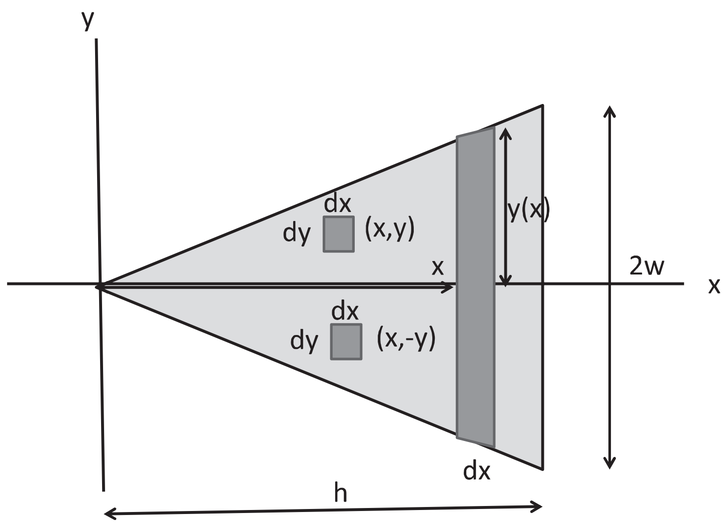

Figure 8.4 The CM calculation of a triangle of base $2w$ and height h. It is viewed as a weighted sum over rods of width dx and height $2y(x)$.

The Triangle in Figure 8.4 has base $2w$, hight h
and a mass per unit area or areal density

$$\rho = \frac{M}{A} = \frac{M}{wh}. \tag{8.33}$$

Where is the center of mass of this object? Again, by symmetry, you can tell that Y, the y coordinate of the center of mass, must be zero. For every tiny square $dxdy$ with some coordinate (x, y), there is a matching one with coordinate $(x, -y)$. For X, you have to do some honest work. We will divide and conquer.

Let us imagine the triangle as composed of thin rectangles of width dx and height $2y(x)$, as indicated. (Each strip is not quite a rectangle, because the edges are slightly tapered, but when $dx \to 0$, they will reduce to rectangles.) The mass dm of the rectangle at a given x is

$$dm = \frac{M}{A} 2y(x)dx = \frac{M}{wh} 2y(x)dx, \tag{8.34}$$

which is just the product of the mass per unit area $\frac{M}{A}$ and the area of the strip $2y(x)dx$. We find $y(x)$ using similar triangles:

$$\frac{y(x)}{w} = \frac{x}{h} \text{ which means } y(x) = \frac{wx}{h}. \tag{8.35}$$

The weighted average of x is then

$$X = \frac{1}{M}\int_{x=0}^{h} \frac{M}{wh} 2y(x)x dx \quad (8.36)$$

$$= \frac{1}{wh}\int_{0}^{h} 2\frac{wx}{h} x dx \quad (8.37)$$

$$= \frac{2}{h^2}\int_{0}^{h} x^2 dx \quad (8.38)$$

$$= \frac{2}{3}h. \quad (8.39)$$

We could have anticipated that X would be skewed to the right, and this formula quantifies that intuition. Note in Eqn. 8.37 that this two-dimensional problem maps onto a one-dimensional one, with a linear density proportional to x, that is, $\rho(x) \propto x$. This is because each vertical strip may be replaced by a point mass on the x-axis proportional to $y(x)$, which in turn grows linearly with x.

To summarize, when we work with extended bodies or more than one body, we can replace the entire body by a single point for certain purposes. The single point is called a center of mass or CM. The CM is fictitious. It has a mass equal to the total mass. It has a location **R** that moves in response to the total external force:

$$M\frac{d^2\mathbf{R}}{dt^2} = \mathbf{F}_e. \quad (8.40)$$

The center of mass is not aware of internal forces, and that's what we want to exploit.

One class of problems has a net external force $\mathbf{F}_e$, and there we know that the CM responds as a point to $\mathbf{F}_e$, regardless of its constituents. For example, a jumbled mass of constituents tossed in the air follows the parabolic trajectory of a point mass, in response to gravity. This is just a one-body problem, which we have studied extensively. So we move on.